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Flexible 4-day workweek scheduling with weekend work frequency constraints

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Abstract

A new integer programming model and a two-stage solution method are presented for the flexible 4-day workweek days-off scheduling problem with weekend work frequency constraints. In this problem, employees are given 3 days off per week, out of which either 2 or 3 must be consecutive. Two alternative constraints are imposed to ensure that employees get a sufficient proportion of weekends off. In the first stage, the dual solution is utilized to determine the minimum workforce size. In the second stage, a constraint specifying the minimum workforce size is appended to the IP model, greatly improving computational efficiency. Moreover, multiple-week rotation schedules are generated to ensure that all conditions are satisfied as employees switch from one work pattern to another in successive weeks.

Keywords: Labor scheduling; Optimization; Integer programming; Staffing

1. Introduction

Labor scheduling problems are classified into three types: (i) shift, or time-of-day, scheduling, (ii) days-off, or day-of-week, scheduling, and (iii) tour scheduling, which combines the first two types. For organizations operating 7 days a week, such as utilities and hospitals, employee days-off scheduling is a significant and relevant problem. Given a set of labor demands for each day of the week, the number of employees assigned to each days-off pattern must be determined. The objective is to minimize the number or cost of the employees, while satisfying daily labor demands and any other labor or financial requirements.

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The most usual type of days-off schedules is referred to as the (5,7) problem, in which each work pattern includes 5 workdays and 2 consecutive off-days per week. Several authors present various approaches to the (5,7) problem, including Baker (1974), Baker and Magazine (1977), Bartholdi, Orlin, and Ratliff (1980), Brownell and Lowerre (1976), Burns (1978), and Vohra (1987). Recently, there has been a lot of interest in alternative work schedules, including 4-day workweeks. This research interest was illustrated by Alfares (2000), Billionnet (1999), Burns, Narasimhan, and Smith (1998), Hung (1991, 1994), Lankford (1998), and Nanda and Browne (1992, pp. 245–249).

This paper combines compressed workweeks with flexible days-off schedules. Each employee is given 3 days off per week, out of which either 2 or 3 days must be consecutive. Moreover, restrictions are imposed to limit the frequency of weekend work assignments. New integer programming (IP) models are developed to represent two alternative constraints on weekend work frequency. A two-stage method utilizing the dual solution and primal–dual relations is presented to obtain either the minimum number or cost of the workforce.

In general, the workforce size is first calculated, and then a workforce size constraint is added to IP formulation of the problem to efficiently obtain optimum integer solutions. Moreover, rotation schedules are constructed over several weeks to guarantee that work patterns in one week fit the work patterns in the following week while satisfying all constraints. The new method is much more computationally efficient than existing IP models. This advantage is significant in applications where many days-off scheduling problems must be solved, such as labor scheduling for projects (Alfares & Bailey, 1997).

Subsequent sections are organized as follows. First, a review of related literature is given. Then, starting with the first type of weekend work frequency constraints, the new IP model is presented. Subsequently, the procedure for determining the minimum workforce size is described, methods for assigning employees to days-off patterns are presented, and computational comparisons with IP are discussed. Alternative weekend work frequency constraints are then analyzed. Finally, examples are solved and conclusions are given.

2. Review of related research

Nanda and Browne (1992) provide the most recent and comprehensive survey of employee days-off scheduling literature. In the past, a lot of attention has been focused on the (5,7) problem, in which two consecutive days off are given per week. For example, Baker and Magazine (1977), Brownell and Lowerre (1976), and Burns (1978) considered different variations of the (5,7) problem with cyclic demands and alternative weekends off. More recently, the focus has shifted toward compressed workweek scheduling, such as the (4,7) problem in which each employee works 4 consecutive days per week. This scheduling alternative is becoming increasingly popular in practice. For example, Lankford (1998) reports a recent implementation of a 4-day workweek schedule at the Analytical Central Call Management Group at Hewlett Packard, noting that cross training is the key to its success.

Hung (1991) analyzes two homogeneous workforce scheduling models under two assumptions: (i) D employees are required on weekdays and E employees on weekends, and (ii) each employee must receive at least A out of B weekends off. In the first model, each employee works 4 days and rests 3 days each week. In the second model, the scheduling cycle is 2 weeks, during which each employee works 4 days and rests 3 days in one week, and works 3 days and rests 4 days in the other week. Under the same assumptions, Narasimhan (1997) considers days-off scheduling for a hierarchical workforce, where each employee cannot be assigned more than 5 consecutive working days.

Hung (1994) develops a heuristic algorithm for hierarchical workforce scheduling under variable labor demand that allows employee substitution. The workweek length can be 3, 4, or 5 days, and the objective is to minimize the workforce cost. To ensure optimality, the heuristic algorithm must be followed by a lengthy branch-and-bound search method. Billionnet (1999) uses IP to formulate and solve the same problem. Burns et al. (1998) present an algorithm for 3-day and 4-day workweeks, with variable daily demand and limits on work stretch lengths.

The algorithm presented in this paper applies to single-shift 4-day workweek scheduling. The algorithm provides optimal solution for the problem under the following assumptions:

(i) the demand for employees may vary from day to day for the given week,
(ii) at least 2 of the 3 weekly off-days are consecutive, and
(iii) two alternative weekend work frequency constraints:
  (I) each employee takes on average a proportion P of *full weekends* off, or
  (II) each employee takes on average a proportion P of *weekend days* off.

3. Integer programming models

As far as the author knows, the model formulated below is the first to represent restrictions on weekend work frequency as integer linear programming constraints. Starting with alternative I weekend work frequency constraints, the most common type found in practice, the flexible 4-day labor scheduling problem can be represented as follows

$$\text{Minimize } W = \sum_{j=1}^{28} x_j \tag{1}$$

subject to

$$\sum_{j=1}^{28} a_{ij} x_j \geq r_i, \quad i = 1, 2, \ldots, 7 \tag{2}$$

$$\frac{\sum_{j \in J_2} x_j}{\sum_{j=1}^{28} x_j} \geq P, \qquad \text{or} \qquad -P\sum_{j \in J_0} x_j - P\sum_{j \in J_1} x_j + (1-P)\sum_{j \in J_2} x_j \geq 0 \tag{3}$$

$$x_{13} + x_{20} + x_{27} \leq MQ, \tag{4}$$

$$\sum_{j \in E \cup F} x_j \geq Q, \quad Q = 0, 1 \tag{5}$$

$$x_j \geq 0 \qquad \text{and integer}, \qquad j = 1, 2, \ldots, 28 \tag{6}$$

where W is the workforce size, i.e. total number of employees assigned to all patterns, x_j is the number of employees assigned to weekly days-off work pattern j,

$$a_{ij} = \begin{cases} 1 & \text{if day } i \text{ is a work day for pattern } j, \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, 2, \ldots, 7).$$

For alternative I:

$$a_{8j} = \begin{cases} -P & \text{if } j \in J_0 \text{ or } j \in J_1, \\ 1 - P & \text{if } j \in J_2. \end{cases}$$

Table 1 shows matrix $A = \{a_{ij}, i = 1, \ldots, 8, j = 1, \ldots, 28\}$. r_i is the minimum number of employees required on day i, $i = 1, 2, \ldots, 7$; $r_8 = 0$; P is average proportion of full weekends off, $0 \le P \le 1$; $Q = 1$ if $x_{13} + x_{20} + x_{27} \ge 1$, $Q = 0$ otherwise; M is any number ≥ 3; E, set of days-off patterns with day 7 off = {5,6,7,12,15,19,23,24,26}; F, set of days-off patterns with day 1 off = {1,6,7,9,14,17,25,28}; J_k, set of days-off patterns with k weekend days off per week, $k = 0, 1, 2$; $J_0 = \{1, 2, 3, 8, 9, 10, 17, 21, 28\}$; $J_1 = \{4, 7, 11, 13, 14, 15, 16, 18, 20, 22, 23, 24, 25, 27\}$; and $J_2 = \{5, 6, 12, 19, 26\}$.

The objective (1) is to minimize the workforce size, i.e., the total number of employees assigned. The right-hand side of expression (2) represents the number of employees required for the given day i, while the left-hand side represents the total number of employees assigned for that day. The denominator on the left-hand side of expression (3) represents the total number of employees assigned for the given weekly schedule, while the numerator represents the number of employees with full weekends off. Constraint (3) can be expressed in the two alternative forms shown, or it can be written in the same form as expression (2), but with $i = 8$.

Constraints (4) and (5) ensure that 2 successive days-off per week are still taken as work patterns are linked from one week to the next during the rotation cycle. Using any rotation scheme, 2 successive days off each week are automatically guaranteed for all days-off patterns except patterns 13, 20, and 27. These three patterns have 1 day off at the start of the week (day 1) and 1 day off at the end of the week (day 7). To have two consecutive days off during rotation, patterns 13, 20, and 27 must be either preceded by a pattern from the set E or followed by a pattern from the set F. Constraints (4) and (5) ensure that if these three patterns are assigned, then at least one pattern from set E or F will be available to precede or follow them.

3. Stage 1: determining the minimum workforce size

Because of the presence of the binary variable Q, it would be difficult to directly analyze the consequence of constraints (4) and (5) on the workforce size W. Therefore, we will instead investigate the effect of including the following stronger (yet simpler) constraint in the IP model:

$$\sum_{j \in E \cup F} x_j \ge x_{13} + x_{20} + x_{27}. \tag{7}$$

Constraint (7) can be expressed in the same form as expression (2), but with $i = 9$. For the flexible 4-day workweek scheduling model defined by expressions (1)–(3) and (6) and (7), the dual model with dual

Table 1
Days-off matrix $A = \{a_{ij}\}$ and cost vector $C = \{c_j\}$ for the 28 days-off work patterns

i	j																											
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	1	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0	1	0	0
2	0	0	1	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0	1	0
3	0	0	0	1	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0	1
4	1	0	0	0	1	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1	0
5	1	1	0	0	0	1	1	0	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1	0	0	1	1	1
6	1	1	1	0	0	0	1	1	1	1	0	0	1	0	1	0	1	0	0	1	1	0	1	1	0	0	1	1
7	1	1	1	1	0	0	0	1	1	1	1	0	0	1	0	1	1	1	0	0	1	1	0	0	1	0	0	1
8	$-P$	$-P$	$-P$	$-P$	$1-P$	$1-P$	$-P$	$-P$	$-P$	$-P$	$-P$	$1-P$	$-P$	$-P$	$-P$	$-P$	$-P$	$-P$	$1-P$	$-P$	$-P$	$-P$	$-P$	$-P$	$-P$	$1-P$	$-P$	$-P$
c_j	$4+2\beta$	$4+2\beta$	$4+2\beta$	$4+\beta$	4	4	$4+\beta$	$4+2\beta$	$4+2\beta$	$4+2\beta$	$4+\beta$	4	$4+\beta$	$4+\beta$	$4+\beta$	$4+\beta$	$4+2\beta$	$4+\beta$	4	$4+\beta$	$4+2\beta$	$4+\beta$	$4+\beta$	$4+\beta$	$4+\beta$	4	$4+\beta$	$4+2\beta$

variables $\{y_1, \dots, y_9\}$ is given as:

$$\text{Maximize } W = \sum_{i=1}^{9} r_i y_i \tag{8}$$

subject to

$$\sum_{i=1}^{9} a_{ij} y_i \le 1, \;\; j = 1, \dots, 28 \tag{9}$$

$$y_i \ge 0, \;\; i = 1, \dots, 9. \tag{10}$$

Given a week's seven daily labor demands $r_1, \dots, r_7$, the minimum workforce size W can be determined from the above dual model using primal–dual relations, without IP. To solve the dual problem we allocate the unit resource—right-hand side of expression (9)—among the dual variables in order to maximize the dual objective W, which is a linear combination of labor demands. Depending on the demands $r_1, \dots, r_7$, there are four possible solutions.

(1) The most obvious solution is to allocate the whole unit resource to the maximum demand, i.e. if $r_k = \max\{r_1, \dots, r_7\}$, set $y_k = 1$ and all other dual variables to zero. In this case $W = r_k = r_{\max}$. This solution is better than or equal to the average that would result from assigning a value of $1/m$ to any m dual variables, where $m = 2, \dots, 9$. However, the structure of the problem makes it possible to assign a value of $1/m$ to *more* than m dual variables. There are three cases where this is possible, which are discussed next.

(2) Excluding y_8 and y_9, each constraint in expression (9) contains only four non-zero dual variables. Therefore, it is possible to divide the unit right-hand side of each constraint among those four variables. Thus it is feasible to assign a value of 1/4 to the first *seven* dual variables: $y_1, y_2, \dots, y_7$ and a value of zero to both y_8 and y_9. In this case the workforce size is equal to $W = \sum r_i/4$.

(3) Another allocation applies to seven sets of four dual variables, whose subscripts, denoted by s_i, are shown in Table 2. Each set $s_i = \{i, i+1, i+3, i+5\}$, $i = 1, \dots, 7$, is a circular set which has a cycle of 7. At least one of any two *adjacent* dual variables y_k, y_{k+1} (corresponding to 2 successive off days) with zero coefficients in constraints (9) must belong to a set $\{y_i, y_{i+1}, y_{i+3}, y_{i+5}\}$. Therefore, no more than three variables from each set have non-zero coefficients in any constraint, and we can assign a value of 1/3 to all *four* variables in a given set. Choosing the set with the maximum total demand, the workforce size is equal to $W = \max\{\sum_{j \in s_i} r_j\}/3 = \max\{S_i/3\}$, $i = 1, 2, \dots, 7$.

To illustrate this point, let us consider for example the set $s_1 = \{1, 2, 4, 6\}$. If we assume only dual variables y_1, y_2, y_4, and y_6 are basic (setting all other dual variables to zero), each dual constraint in expression (9) will have no more than three variables on the left-hand side. For example, the first five dual constraints in expression (9) become:

$$j = 1: y_4 + y_6 \le 1 \qquad j = 2: y_1 + y_6 \le 1 \qquad j = 3: y_1 + y_2 + y_6 \le 1 \qquad j = 4: y_1 + y_2 \le 1$$

$$j = 5: y_1 + y_2 + y_4 \le 1.$$

Since the objective is to maximize $W = \sum r_i y_i$, the optimum dual solution is given by:

$$y_1 = y_2 = y_4 = y_6 = 1/3 \qquad W = (r_1 + r_2 + r_4 + r_6)/3 = S_1/3.$$

Table 2
Sets of subscripts s_i defined by Eq. (13)

i	s_i
1	1,2,4,6
2	2,3,5,7
3	1,3,4,6
4	2,4,5,7
5	1,3,5,6
6	2,4,6,7
7	1,3,5,7

(4) Finally, another dual solution is obtained if we set all first five dual variables to zero ($y_i = 0$, $i = 1, \ldots, 5$). The first seven rows in dual constraints in expression (9) become:

$$j = 1 : y_6 + y_7 - Py_8 + y_9 \leq 1 \qquad j = 2,3 : y_6 + y_7 - Py_8 \leq 1 \qquad j = 4 : y_7 - Py_8 \leq 1$$

$$j = 5,6 : (1 - P)y_8 + y_9 \leq 1 \qquad j = 7 : y_6 - Py_8 + y_9 \leq 1.$$

If we assume all above constraints are equations, the solution is given by:

$$(y_6 + y_7) = y_8 = 1/(1 - P), \qquad y_9 = 0 \qquad W = \max(r_6, r_7)/(1 - P).$$

It is also possible to assign a value of 1/4 to five or six dual variables. However, these allocations are obviously dominated by case 2 above. To determine W, we choose the maximum value obtained from the four above cases, and round it up to the nearest integer. Rounding up non-integer values of W to reach feasible integer solutions is a technique used by several authors, such as Baker (1974), Bartholdi et al. (1980), Burns (1978), and Vohra (1987). Moreover, this technique will be empirically shown to produce feasible integer solutions for 202 test problems. Therefore, we obtain the following expression for the minimum W

$$W = \max\left\{ r_{\max}, \left\lceil \frac{1}{4}\sum_{i=1}^{7} r_i \right\rceil, \left\lceil \frac{S_{\max}}{3} \right\rceil, \left\lceil \frac{\max(r_6, r_7)}{1 - P} \right\rceil \right\} \tag{11}$$

where $r_{\max} = \max\{r_1, r_2, \ldots, r_7\}$, $S_{\max} = \max\{S_1, S_2, \ldots, S_7\}$, $\lceil a \rceil =$ smallest integer $\geq a$, and

$$S_i = \sum_{j \in S_i} r_i, \quad i = 1, 2, \ldots, 7 \tag{12}$$

$$s_i = \{i, i+1, i+3, i+5\}, \quad i = 1, 2, \ldots, 7 \tag{13}$$

where s_i is circular set with a cycle $= 7$ (see Table 2).

3.1. Interpretation of the bounds on workforce size W

It seems appropriate at this point to interpret the economic meaning of the four bounds in Eq. (11). First, the workforce size must be greater than the number of employees required on any given day; thus

$W \geq r_{\max}$. Second, since each employee is assigned 4 workdays per week, the total man-days assigned is $4\sum x_j = 4W$. This has to be greater than the total man-days required, which is $\sum r_i$. Thus, $4W \geq \sum r_i$, or $W \geq \sum r_i/4$.

For the third bound, let us take $S_1 = (r_1 + r_2 + r_4 + r_6)$ as an example. If we sum rows 1, 2, 4, and 6 in constraints (2) referring to matrix A in Table 1 and derive an upper bound for it, we obtain:

$$3\sum x_j \geq r_1 + r_2 + r_4 + r_6. \tag{14}$$

Thus, in order to satisfy labor demands on days 1, 2, 4, and 6, $W \geq S_1/3$. A similar result will be obtained for sets $S_2, \ldots, S_7$, thus leading to the third bound ($W \geq S_{\max}/3$). It must be noted that this third bound is the cost of insisting on having 2 consecutive days off. If non-consecutive days off where allowed, it would be possible to assign employees to a days-off pattern that is, for example, off on days 3, 5, and 7 (working on days 1, 2, 4, and 6). In that case, labor demands on days 1, 2, 4, and 6 could be satisfied by assigning employees to work on these 4 days, and the number of employees required would be equal to maximum demand during the 4 days, i.e. $W \geq \max(r_1, r_2, r_4, r_6)$, which is dominated by the first bound ($W \geq r_{\max}$).

For the fourth bound, we resort to the dual solution. Since either y_6 and y_8 or y_7 and y_8 are basic, then either primal constraints 6 and 8 or 7 and 8 can be tight. Assuming constraints 6 and 8 are tight, let us sum rows 6 and 8 in constraints (2), obtaining

$$(1-P)\sum_{j\in J_0} x_j - P\sum_{j\in J_{1a}} x_j + (1-P)\sum_{j\in J_{1b}} x_j + (1-P)\sum_{j\in J_2} x_j \geq r_6 \tag{15}$$

where J_{1a} is the subset of J_1 patterns with only day 6 off = {4,11,14,16,18,22,25}, J_{1b} is the subset of J_1 patterns with only day 7 off = {7,13,15,20,23,24,27}.

However, if constraints 6 and 8 are tight, then $x_j = 0$ for all $J \in J_{1a}$, and expression (15) reduces to: $(1-P)W \geq r_6$, or $W \geq r_6/(1-P)$. Using the same procedure, if we sum constraints 7 and 8 assuming both are tight, then $x_j = 0$ for all $J \in J_{1b}$, and we obtain: $W \geq r_7/(1-P)$. Thus, in order to satisfy demands on both days 6 and 7 (weekend), we must combine these two results together, obtaining: $W \geq \max(r_6, r_7)/(1-P)$. This fourth bound corresponds to the cost of giving employees a proportion P of weekends off.

4. Stage 2: assigning employees to days-off patterns

4.1. Minimum cost assignment

After determining the workforce size W by Eq. (11), the objective at this stage is to assign the W employees to different days-off patterns in order to minimize total cost. Naturally, the cost of each days-off pattern is related to the number of overtime-paid weekend workdays. The 28 columns of matrix A represent days-off work patterns, while the first seven rows represent days of the week, with rows 6 and 7 corresponding to the weekend. Assuming each employee is paid 1 unit per regular workday and $1+\beta$ units ($\beta \geq 0$) per weekend workday, the weekly costs of the 28 days-off patterns $\{c_1, c_2, \ldots, c_{28}\}$ are shown in Table 1. Changing the objective to that of minimizing total cost, Eq. (1) is replaced by

$$\text{Minimize } Z = \sum_{j=1}^{28} c_j x_j \tag{16}$$

where c_j is the weekly cost of days-off pattern j per employee, shown in Table 1.

Introducing these costs in the dual model, the right-hand side of dual constraints (9) changes from 1 to c_j. Naturally, the dual solutions slightly differ from those discussed above with unit costs. However, in all four cases, the workforce size W does not change. This means that for the cost structure defined by $\{c_1, c_2, \ldots, c_{28}\}$, for all $\beta \geq 0$, the minimum *cost* is always obtained with the minimum *number* of employees. In other words, introducing the varying costs for different days-off patterns defined by $c_1, \ldots, c_{28}$ does not affect the workforce size.

4.2. Solution procedure

Given $r_1, \ldots, r_7$, β, and P, the value of the minimum workforce size W is first calculated by Eq. (11), then the following constraint is added to the primal IP model defined by expressions (2)–(6) and (16)

$$\sum_{j=1}^{28} x_j = W. \tag{17}$$

4.3. Rotation scheme

Constraints included in the model specify that each employee must receive: (i) a proportion P of weekends off, and (ii) 2 successive days off per week. A rotation scheme is now introduced to ensure that both constraints are satisfied as employees switch from one pattern to another over a multiple-week rotation period. The solution of the IP model specifies the number of employees assigned to each days-off pattern $(x_1, \ldots, x_{28})$ and the total workforce W $(x_1 + \cdots + x_{28})$ for a single week. This solution provides the basis for our rotation scheme presented below.

First, we define a W-week rotation cycle, during which each employee is assigned to each pattern j for a period of x_j weeks ($j = 1, \ldots, 28$). For feasibility and fairness, all W employees must go through the same sequence of assignments to patterns over this W-week rotation cycle, but each employee starts the sequence at a different week of the cycle. To guarantee a feasible sequence, assignments to patterns 13, 20, and 27 must be consecutive, and must be either immediately preceded by a pattern from the set E or immediately followed by a pattern from the set F. Constraints (4) and (5) ensure that at least one pattern from either set E or F is assigned if patterns 13, 20, and 27 are active.

4.4. Computational results

The addition of workforce size constraint (17) has been found to drastically reduce IP computation times. In order to evaluate the effect of adding constraint (17), computational experiments were carried out using 202 test problems. In all these problems, the value of β was set at 0.5 to indicate 50% higher pay for weekend work. Moreover, the value of P was set at 1/2 to indicate a requirement of every other weekend off. This is a typical value of P used by many researchers such as Baker and Magazine (1977), Brownell and Lowerre (1976), and Burns (1978).

The 202 test problems, described by Alfares (1998), are divided into 10 sets with different demand types, but all have an average demand of 50 employees per day. The first six sets involve 17 problems each, with a demand range of 34–64, and have specific labor demand patterns: level, trend, concave, convex, unimodal, and sinusoidal. The last four sets involve 25 problems each, with labor demands randomly distributed over the intervals: [34,66], [0,100], [20,80], and [45,55].

Microsoft Excel Solver®, on an 866-MHz Pentium III PC, was used to solve the test problems by IP. The results of computational experiments are summarized in Table 3. For all 10 problem sets, adding constraint (17) has decreased the average, maximum, and variation of solution times. Overall, average solution time dropped by 92%, from 1.84 to 0.15 s. Maximum solution time fell by 99.8%, from 337.41 to only 0.77 s. Similarly, the standard deviation decreased from 23.73 to only 0.06 s. Clearly, adding constraint (17) leads to a remarkable reduction in both the mean and variation of solution times.

5. Alternative weekend-off frequency constraints

In the foregoing presentation, weekend-off frequency constraints specified a minimum proportion P of full weekends off for each employee. Suppose now that weekend frequency constraints are modified to include *half weekends* off in addition to full weekends off in the proportion P. To incorporate this modification, the following changes have to be made in the model.

First, constraint (3) must be replaced by the following constraint, which ensures that each employee takes a proportion P of *weekend days off*

$$\frac{\sum_{j\in J_1} x_j + 2\sum_{j\in J_2} x_j}{2\sum_{j=1}^{28} x_j} \geq P \qquad \text{or} \qquad -2P\sum_{j\in J_0} x_j + (1-2P)\sum_{j\in J_1} x_j + (2-2P)\sum_{j\in J_2} x_j \geq 0. \tag{18}$$

For weekend work frequency constraints alternative II, we must redefine the constraints coefficients as follows:

$$a_{8j} = \begin{cases} -2 & \text{if } j \in J_0, \\ 1-2P & \text{if } j \in J_1, \\ 2-2P & \text{if } j \in J_2. \end{cases}$$

The numerator on the left-hand side of expression (18) represents the number of weekend man-days assigned off in the given weekly schedule, while the denominator represents the maximum number of weekend man-days (2 days off per employee). In order to determine the workforce size W, the dual of the IP model corresponding to alternative II must be solved. The first three dual solutions, i.e., first three bounds in Eq. (11), are the same as for alternative I. However, the fourth dual solution changes as

Table 3
Integer programming solution times in seconds (minimum, average, maximum, and standard deviations) with and without constraint (17)

Weekend work frequency constraints	Without constraint (17)				With constraint (17)			
Alternative	Min	Ave	Max	Std dev	Min	Ave	Max	Std dev
I	0.11	1.84	337.41	23.73	0.11	0.15	0.77	0.06
II	0.11	1.19	80.30	7.59	0.16	0.17	0.22	0.02

follows. If we set $y_i = 0$, $i = 1, \ldots, 5$, the first seven rows in dual constraints (9) become:

$$j = 1 : y_6 + y_7 - 2Py_8 + y_9 \le 1 \quad j = 2,3 : y_6 + y_7 - 2Py_8 \le 1 \quad j = 4 : y_7 + (1 - 2P)y_8 \le 1$$

$$j = 5,6 : (2 - 2P)y_8 + y_9 \le 1 \quad j = 7 : y_6 + (1 - 2P)y_8 + y_9 \le 1.$$

If we assume all above constraints are equations, the solution is given by:

$$y_6 = y_7 = y_8 = 1/(2 - 2P), \qquad y_9 = 0 \qquad W = (r_6 + r_7)/(2 - 2P).$$

To interpret this alternative fourth bound on W, we simply need to sum rows 6 and 7 (weekend labor requirements) in constraints (2) plus new weekend off frequency constraints (18), obtaining:

$$(2 - 2P)\sum_{j=1}^{28} x_j \ge r_6 + r_7. \tag{19}$$

Thus in order to satisfy demands on days 6 and 7, $W \ge (r_6 + r_7)/(2 - 2P)$. Therefore, Eq. (11) is replaced by the following expression for the minimum workforce size W

$$W = \max\left\{ r_{\max}, \left\lceil \frac{1}{4}\sum_{i=1}^{7} r_i \right\rceil, \left\lceil \frac{S_{\max}}{3} \right\rceil, \left\lceil \frac{r_6 + r_7}{2 - 2P} \right\rceil \right\}. \tag{20}$$

The solution procedure now consists of calculating the workforce size W by Eq. (20), and then adding expression (17) to the modified IP model defined by expressions (2), (4)–(6), and (16). The same rotation scheme involving a cycle of W weeks must again be used for the W employees. Table 3 summarizes results of experiments to determine the effect of adding constraint (17) to the modified model, using the 202 test problems previously described. Overall, average solution time dropped by 86%, from 1.19 to 0.17 s. Maximum solution time fell by 99.7%, from 80.30 to only 0.22 s. Similarly, the standard deviation decreased from 7.59 to only 0.02 s.

6. Solved examples

The two following examples are used to illustrate the simple calculations required for applying the proposed solution method. Example 1 corresponds to alternative I of weekend work frequency constraints, while Example 2 corresponds to alternative II.

6.1. Example 1

Assume that the required proportion of full weekends off P is given as 0.5, and that the daily labor demands for a certain work week are given as:

$$r_1, r_2, \ldots, r_7 = 8, 8, 2, 8, 2, 7, 5.$$

Using Eq. (12), the following values are calculated:

$$S_1, S_2, \ldots, S_7 = 31, 17, 25, 23, 19, 28, 17.$$

Hence

$$r_{\max} = 8 \qquad \sum r_i/4 = 40/4 = 10 \qquad S_{\max}/3 = S_1/3 = 31/3 = 10.33$$

$$\max(r_6, r_7)/(1 - P) = 7/0.5 = 14.$$

Using Eq. (11),

$$W = \max\{8, 10, \lceil 10.33 \rceil, 14\} = 14.$$

Without requiring 2 days off to be consecutive and without requiring a proportion of weekends off, W would be the maximum of $r_{\max}$ and $\sum r_i/4$ (rounded up), or 10. Including only the requirement of consecutive pairs of days off, the workforce size becomes: $W = \lceil S_1/3 \rceil = 11$. Thus the cost of consecutivity is one extra employee. Adding also the requirement of 50% of weekends off increases W to 14, hence the cost of this requirement is three extra employees. To the model defined by expressions (2)–(6) and (16), we add the constraint defined by Eq. (17): $\sum_{j=1}^{28} x_j = 14$ and solve by IP to obtain the days-off assignments $x_1, \ldots, x_{28}$. The IP solution is given by:

J_0 and J_1 patterns: $x_8 = 5$, $x_{13} = 2$
J_2 (full weekend off) patterns: $x_5 = 5$, $x_6 = 1$, $x_{26} = 1$.

A 14-week rotation cycle must be used, during which each employee is assigned 5 weeks to pattern 8, 2 weeks to pattern 13, 5 weeks to patterns 5, and so on. To maintain 2 successive days off per week, pattern 13 assignments must be consecutive, and either immediately followed by pattern 6 or immediately preceded by pattern 5, 6, or 26. A 14-week feasible rotation cycle is shown in Table 4, in which pattern 13 assignments are directly followed by a pattern 6 assignment.

6.2. Example 2

Assume the same daily labor demands as given in Example 1, but now the specified value of $P = 0.5$ pertains to the required proportion of weekend days off. Since the first three bounds on W are the same for Eqs. (11) and (20), only the fourth bound needs to be calculated as follows:

$$(r_6 + r_7)/(1 - P) = (5 + 7)/2(0.5) = 12.$$

Using Eq. (20),

$$W = \max\{8, 10, \lceil 10.33 \rceil, 12\} = 12.$$

Table 4
A cyclic 14-week rotation schedule for example 1, where bold cells represent an assignment to a J_2 (full-weekend off pattern)

Employee	Week													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	**5**	8	8	8	8	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**
2	**5**	**5**	8	8	8	8	8	13	13	**6**	**26**	**5**	**5**	**5**
3	**5**	**5**	**5**	8	8	8	8	8	13	13	**6**	**26**	**5**	**5**
4	**5**	**5**	**5**	**5**	8	8	8	8	8	13	13	**6**	**26**	**5**
5	**5**	**5**	**5**	**5**	**5**	8	8	8	8	8	13	13	**5**	**26**
6	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8	8	8	13	13	**6**
7	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8	8	8	13	13
8	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8	8	8	13
9	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8	8	8
10	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8	8
11	8	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8	8
12	8	8	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8	8
13	8	8	8	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**	8
14	8	8	8	8	8	13	13	**6**	**26**	**5**	**5**	**5**	**5**	**5**

To the model defined by expressions (2), (4)–(6), and (16), we add the constraint: $\sum_{j=1}^{28} x_j = 12$, and solve by IP to obtain the days-off assignments $x_1, \ldots, x_{28}$:

J_0 patterns: $x_1 = 2$, $x_3 = 3$
J_1 patterns: $x_{13} = 1$, $x_{23} = 1$
J_2 patterns: $x_6 = 1$, $x_{19} = 4$.

Using a 12-week rotation cycle, pattern 13 assignments must be either preceded by pattern 6, 19, or 23, or followed by pattern 1 or 6.

7. Conclusions

A new optimization algorithm for the flexible (4,7) labor days-off scheduling problem with weekend work frequency constraints has been presented. In this problem, employees are given 3 off days per week, out of which either 2 or 3 days must be consecutive, and a certain proportion of weekends must be off. New IP models have been developed to represent two alternative restrictions on weekend work frequency. For each alternative, an expression has been derived from the dual solution to determine the minimum workforce size. Cyclic multiple-week rotation schedules have been developed for assigning employees to work patterns fairly while satisfying all constraints.

For the two alternative weekend work frequency constraints, the addition of a workforce-size constraint to the IP model has been proposed to improve computational efficiency. Based on computational experiments with 202 test problems, the appended model was found to be on average nine times faster than the traditional IP model. This advantage is especially significant in applications where a large number of these models must be solved.

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