COE 202: Digital Logic Design
Number Systems
Part 1

Dr. Ahmad Almulhem
Email: ahmadsm AT kfupm
Phone: 860-7554
Office: 22-324
Objectives

1. Weighted (positional) number systems
2. Features of weighted number systems.
3. Commonly used number systems
4. Important properties
Introduction

• A **number system** is a set of **numbers** together with one or more **operations** (e.g. add, subtract).

• Before digital computers, the only known number system is the **decimal number system** (النظام العشري)
  – It has a total of ten digits: {0,1,2,…..,9}

• From the previous lecture:
  – Digital systems deal with the binary system of numbering i.e. only 0’s and 1’s
  – Binary system has more reliability than decimal

• All these numbering systems are also referred to as **weighted numbering systems**
Weighted Number System

A number $D$ consists of $n$ digits and each digit has a position.
Every digit position is associated with a fixed weight.
If the weight associated with the $i$th position is $w_i$, then the value of $D$ is given by:

$$D = d_{n-1}w_{n-1} + d_{n-2}w_{n-2} + \ldots + d_1w_1 + d_0w_0$$

Also called positional number system
### Example

The Decimal number system is a weighted number system. For Integer decimal numbers, the weight of the rightmost digit (at position 0) is 1, the weight of position 1 digit is 10, that of position 2 digit is 100, position 3 is 1000, etc.

<table>
<thead>
<tr>
<th>Position</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>9</td>
<td>3</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Weight</td>
<td>1000</td>
<td>100</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>Value</td>
<td>$9 \times 1000$</td>
<td>$3 \times 100$</td>
<td>$7 \times 10$</td>
<td>$5 \times 1$</td>
</tr>
<tr>
<td>Value</td>
<td>9000</td>
<td>300</td>
<td>70</td>
<td>5</td>
</tr>
</tbody>
</table>
The Radix (Base)

- A digit $d_n$ has a weight which is a power of some constant value called **radix (r)** or **base** such that $w_i = r^i$.
- A number system of radix $r$, has $r$ allowed digits $\{0, 1, \ldots (r-1)\}$
- The leftmost digit has the highest weight and called **Most Significant Digit (MSD)**
- The rightmost digit has the lowest weight and called **Least Significant Digit (LSD)**
Example

- Decimal Number System
- Radix (base) = 10
- \( w_i = r^i \), so
  - \( w_0 = 10^0 = 1 \),
  - \( w_1 = 10^1 = 10 \)
  - \( \ldots \)
  - \( w_n = r^n \)
- Only 10 allowed digits: \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \)

\[
9375 = 5 \times 10^0 + 7 \times 10^1 + 3 \times 10^2 + 9 \times 10^3 \\
= 5 \times 10^3 + 7 \times 10^1 + 3 \times 10^0 + 9 \times 10^3
\]

<table>
<thead>
<tr>
<th>Position</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>1000</td>
<td>100</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\text{MSD:} \\
\text{LSD:}
\]
Fractions (Radix point)

- A number D has \( n \) integral digits and \( m \) fractional digits.
- Digits to the left of the radix point (integral digits) have positive position indices, while digits to the right of the radix point (fractional digits) have negative position indices.
- The weight for a digit position \( i \) is given by \( w_i = r^i \).
Example

- For D = 57.6528
  - \( n = 2 \)
  - \( m = 4 \)
  - \( r = 10 \) (decimal number)
- The weighted representation for D is:

\[
\begin{align*}
  i = -4 & \quad d_i r^i = 8 \times 10^{-4} \\
  i = -3 & \quad d_i r^i = 2 \times 10^{-3} \\
  i = -2 & \quad d_i r^i = 5 \times 10^{-2} \\
  i = -1 & \quad d_i r^i = 6 \times 10^{-1} \\
  i = 0 & \quad d_i r^i = 7 \times 10^{0} \\
  i = 1 & \quad d_i r^i = 5 \times 10^{1}
\end{align*}
\]

\[
D = 5 \times 10^1 + 2 \times 10^0 + 9 \times 10^{-1} + 4 \times 10^{-2} + 6 \times 10^{-3} \quad 0.04
\]
Notation

A number D with base $r$ can be denoted as $(D)_r$.
Decimal number 128 can be written as $(128)_{10}$
Similarly a binary number is written as $(10011)_2$

Question: Are these valid numbers?

- $(9478)_{10}$
- $(1289)_2$
- $(111000)_2$
- $(55)_{5}$
Common Number Systems

- Decimal Number System (base-10)
- Binary Number System (base-2)
- Octal Number System (base-8)
- Hexadecimal Number System (base-16)
Binary Number System (base-2)

- $r = 2$
- Two allowed digits \{0,1\}
- A Binary Digit is referred to as bit
- Examples: 1100111, 01, 0001, 11110
- The left most bit is called the Most Significant Bit (MSB)
- The rightmost bit is called the Least Significant Bit (LSB)
Binary Number System (base-2)

- The decimal equivalent of a binary number can be found by expanding the number into a power series:

**Example**

- \( (1 0 1)_2 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 \)
- \[ = 1 \times 1 + 0 \times 2 + 1 \times 4 \]
- \[ = (5)_{10} \]

**Question:**
What is the decimal equivalent of \((110.11)_2\)?
Binary Number System (base-2)

- The decimal equivalent of a binary number can be found by expanding the number into a power series:

  **Example**

  \[ (101)_2 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 \]
  \[ = 1 \times 1 + 0 \times 2 + 1 \times 4 \]
  \[ = (5)_{10} \]

  **Question:**
  What is the decimal equivalent of \((110.11)_2\) ?

  **Answer:** \((6.75)_{10}\)
Octal Number System (base-8)

- $r = 8$
- Eight allowed digits \{0,1,2,3,4,5,6,7\}
- Useful to represent binary numbers indirectly
  - Octal and binary are nicely related; i.e. $8 = 2^3$
  - Each octal digit represents 3 binary digits (bits)
- Example: \((101)_2 = (5)_8\)
- Getting the decimal equivalent is as usual

Example:
\[
(375)_8 = 5 \times 8^0 + 7 \times 8^1 + 3 \times 8^2 \\
= 5 \times 1 + 7 \times 8 + 3 \times 64 \\
= (253)_{10}
\]
Hexadecimal Number System (base-16)

- $r = 16$
- 16 allowed digits \{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}
- Useful to represent binary numbers indirectly
  - Hex and binary are nicely related; i.e. $16 = 2^4$
  - Each hex digit represents 4 binary digits (bits)
- Example: $(1010)_2 = (A)_{16}$
- Getting the decimal equivalent is as usual

Example: $\left(3.B.C\right)_{16} = C \times 16^{-1} + B \times 16^0 + 3 \times 16^1$

Question:

$(9E1)_{16} = (?)_{10}$

Ahmad Almulhem, KFUPM 2010
Hexadecimal Number System (base-16)

- \( r = 16 \)
- 16 allowed digits \{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}
- Useful to represent binary numbers indirectly
  - Hex and binary are nicely related; i.e. \( 16 = 2^4 \)
  - Each hex digit represents 4 binary digits (bits)
- Example: \((1010)_2 = (A)_{16}\)

- Getting the decimal equivalent is as usual

**Example**

\[(3B.C)_{16} = C \times 16^{-1} + B \times 16^0 + 3 \times 16^1
= 12 \times 16^{-1} + 11 \times 16^0 + 3 \times 16
= (59.75)_{10}\]

**Question:**

\[(9E1)_{16} = (?)_{10}
= 1 \times 16^0 + E \times 16^1 + 9 \times 16^2
= 1 \times 1 + 14 \times 16 + 9 \times 256
= (2529)_{10}\]
**Examples**

**Question:** What is the result of adding 1 to the largest digit of some number system?

- \((9)_{10} + 1 = (10)_{10}\)
- \((7)_8 + 1 = (10)_8\)
- \((1)_2 + 1 = (10)_2\)
- \((F)_{16} + 1 = (10)_{16}\)

**Conclusion:** Adding 1 to the largest digit in any number system always has a result of \((10)\) in that number system.

**OCTAL System**

\[
\begin{array}{c}
7 \\
+ \\
1 \\
\hline
8 \\
\hline
\end{array}
\]

\(8\) is an illegal octal digit.

\[
10 = 0\times8^0 + 1\times8^1
\]
Examples

Question: What is the largest value representable using 3 integral digits?

Answer: The largest value results when all 3 positions are filled with the largest digit in the number system.

- For the decimal system, it is $(999)_{10}$
- For the octal system, it is $(777)_8$
- For the hex system, it is $(FFF)_{16}$
- For the binary system, it is $(111)_2$
Examples

Question: What is the result of adding 1 to the largest 3-digit number?

- For the decimal system, \((1)_{10} + (999)_{10} = (1000)_{10} = (10^3)_{10}\)
- For the octal system, \((1)_8 + (777)_8 = (1000)_8 = (8^3)_{10}\)

In general, for a number system of radix \(r\), adding 1 to the largest \(n\)-digit number = \(r^n\)

Accordingly, the value of largest \(n\)-digit number = \(r^n - 1\)

Ahmad Almulhem, KFUPM 2010
Important Properties

• The number of possible digits in any number system with radix $r$ equals $r$.

• The smallest digit is 0 and the largest digit has a value $(r - 1)$
  - Example: Octal system, $r = 8$, smallest digit = 0, largest digit = $8 - 1 = 7$

• The Largest value that can be expressed in $n$ integral digits is $(r^n - 1)$
  - Example: $n = 3$, $r = 10$, largest value = $10^3 - 1 = 999$
Important Properties

• The Largest value that can be expressed in \(m\) fractional digits is \((1 - r^{-m})\)
  – Example: \(n=3, r = 10\), largest value = \(1 - 10^{-3} = 0.999\)

• Largest value that can be expressed in \(n\) integral digits and \(m\) fractional digits is equal to \((r^n - r^{-m})\)

• Total number of values (patterns) representable in \(n\) digits is \(r^n\)
  – Example: \(r = 2, n = 5\) will generate 32 possible unique combinations of binary digits such as \((00000 \rightarrow 11111)\)
  – Question: What about Intel 32-bit & 64-bit processors?
Conclusions

- A weighted (positional) number system has a radix (base) and each digit has a position and weight.
- Commonly used number systems are decimal, binary, octal, hexadecimal.
- A number D with base $r$ can be denoted as $(D)_r$.
- To convert from base-$r$ to decimal, use

$$ (D)_r = \sum_{i=-m}^{n-1} d_i r^i $$

- Weighted (positional) number systems have several important properties.