A Generalization of Hardy's Inequality for Fourier Series

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To whom I love with all the strength of love that she taught me

The love that sustains in the storms of life and protect me

To whom I see in every ray of light in a glorious sunrise

To my mother for whom it is worth living this life.

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Abstract

The following generalization of Hardy's inequality is due to I. Klemes [6] (1993);

"There is a constant $c > 0$ such that for any function $f \in L_1(\mathbb{T})$,

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(n)|^{2}\right)^{1/2} \leq c\|f\|_1 + c\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(-n)|^{2}\right)^{1/2}."$$

The proof was based on an elegant construction, (L. Pigno and B. Smith [11]), of a certain bounded function whose Fourier coefficients have desired properties.

The chief object of this thesis is to record another proof of the above result by using the construction that was originally used to prove the Littlewood conjecture [8]. In addition, a proof is given that the same generalization is equivalent to another one involving the norm of the Besov space $B_{2\,1}^{-1/2}$.

Résumé

La généralisation de l'inégalité de Hardy obtenue par I. Klemes [6] (1993), est donnée par

"Il existe une constante $c > 0$ telle que pour toute fonction $f \in L_1(\mathbb{T})$,

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(n)|^{2}\right)^{1/2}\leq c\|f\|_{1}+c\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(-n)|^{2}\right)^{1/2}."$$

La démonstration de ce résultat repose sur l'élégante construction, (L. Pigno et B. Smith [11]), d'une certaine fonction bornée dont les coefficients de Fourier ont des proprietés désirées.

L'objectif de cette thèse est de donner une autre démonstration du précédent résultat utilisant cette fois une construction intervenant dans la démonstration de la conjecture de Littlewood [8]. De plus, une équivalence entre l'inégalité ci-dessus et une inégalité impliquant la norme de l'espace $B_{2\,1}^{-1/2}$ de Besov est donnée.

Contents

Introduction

In 1927, G. H. Hardy proved in [3] that there is a constant $c > 0$ such that for any function $f \in H_1(\mathbb{T})$,

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1.$$

Hardy's inequality, however, is not true for all functions $f \in L_1(\mathbb{T})$. A simple counter example of that is the Fejér kernel. In view of the foregoing, many mathematicians are trying to generalize it for the whole space $L_1(\mathbb{T})$. For instance, I. Klemes in [6], where he was referring to [13]*, remarked that the following generalization is a well-known open problem:

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c\|f\|_1 + \sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|}{n}, \qquad f \in L_1(\mathbb{T}). \tag{0.1}$$

To the best of our knowledge, this is still an unsolved conjecture, at the time of writing.

On the other hand, many mathematicians succeeded to generalize the inequality in other ways. As a good example, in 1981, the Littlewood conjecture, concerning the L_1-norm of exponential sums, was proved in [8] as a consequence of a special generalization of Hardy's inequality. The proof of that generalization was based on a very clever construction of certain bounded functions with desired Fourier coefficients. Two years later, J. J. F. Fournier [2] gave three other constructions that play the same role as the one given in [8].

In 1993, I. Klemes [6] proved, by using one of those constructions, what we call here the *mixed-norm generalization of Hardy's inequality.* It says: *"There is a constant*

*V. Peller and S. Khrushchev in [10, §3.6] is an earlier reference to this conjecture.

$c > 0$ such that for any function $f \in L_1(\mathbb{T})$,

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^j}|\hat{f}(n)|^2\right)^{1/2} \leq c\|f\|_1 + c\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^j}|\hat{f}(-n)|^2\right)^{1/2}.\text{''}$$

He also remarked that, with some modifications, the other three constructions (including the one in [8]) would also work there.

In chapter 2 of this thesis, we have "re-proved" the mixed-norm generalization of Hardy's inequality by using the construction established in [8], after some modifications of course. There, we have proved also that the same generalization is equivalent to another one involving the norm of the Besov space $B_{2\,1}^{-1/2}$. Then, we have found that there is a similarity between the mixed-norm generalization of Hardy's inequality and a theorem due to V. Peller and S. Khrushchev [10] in the reconstruction problem on Besov spaces (in case $p = 2$ only).

Chapter 1 contains a brief survey of two famous generalizations of Hardy's inequality that are "related" to the thesis' subject; one of them has been already mentioned in this introduction. In addition, we have stated the construction used in [8] and the other three constructions given in [2], for the sake of completeness.

Chapter 0 is devoted to recall and explain shortly some of the basic facts and the preliminaries that we have used in this thesis.

We should note here that throughout this thesis all the variables and indices are assumed to be integers unless otherwise stated.

List of Symbols

$\mathbb{R}$	The set of real numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{N}$	The set of positive integers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{T}$	The unit circle.
$\emptyset$	The empty set.
$\mathbf{Re}z$	The real part of z.
$\mathbf{Im}z$	The imaginary part of z.
$\bar{z}$	The complex conjugate of z.
$[n, m]$	$= \{k \in \mathbb{Z} \ : n \leq k \leq m\}$.
I_j	$[4^{j-1}, 4^j)$.
${}^{-}I_j$	$(-4^j, -4^{j-1}]$.
$\ddot{I}_j$	$[2^{j-1}, 2^j)$.
$\ddot{\ddot{I}}_j$	$[4^{j-1}, 4^j)$.
$f * g$	The convolution of f and g.
$\hat{f}(n)$	The Fourier coefficient of f at n.
$\mathrm{spec}(f)$	$= \{n \in \mathbb{Z} \ : \hat{f}(n) \neq 0\}$.
K_N	The Fejér kernel of order N.
$K_{(\ell)}$	$= K_{2 \cdot 4^{\ell} - 3}$.
$L_p(\mathbb{T})$	The set of all p-integrable functions on $\mathbb{T}$.
$H_p(\mathbb{T})$	$= \{f \in L_p(\mathbb{T}) \ : \hat{f}(n) = 0 \quad \forall\ n < 0\}$.

Chapter 0

Preliminaries

Many basic facts and well-known concepts are used throughout this thesis. As a preparation, we explain them briefly in this chapter. Proofs and more detailed accounts of these facts can be found in [5], [12] or [1].

Terminologies

As usual, let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the *unit circle*. The functions on $\mathbb{T}$ are identified with the 2π-periodic functions on $\mathbb{R}$; hence, the Lebesgue measure on $\mathbb{T}$ can be defined by means of this identification.

The most important property of the Lebesgue measure on $\mathbb{T}$ is its translation invariance, i.e.

$$\int_{\mathbb{T}} f(t-\tau)\, dt = \int_{\mathbb{T}} f(t)\, dt, \quad \tau \in \mathbb{T}.$$

Also the Lebesgue measure on $\mathbb{T}$ is a finite measure with total mass equal to 2π.

L_p-spaces on $\mathbb{T}$.

For $1 \leq p < \infty$, and function f on $\mathbb{T}$, set

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^p\, dt\right)^{1/p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\, dx\right)^{1/p}.$$

The L_p*-space* on $\mathbb{T}$, denoted by $L_p(\mathbb{T})$, is the set of all complex-valued functions f on $\mathbb{T}$ such that $\|f\|_p < \infty$. Furthermore, the $L_\infty(\mathbb{T})$ is defined to be the set of all bounded complex-valued functions on $\mathbb{T}$ with the norm

$$\|f\|_\infty = \sup_{t\in\mathbb{T}} |f(t)|.$$

It is well-known that $L_p(\mathbb{T}),\ 1 \le p \le \infty$, with the norm defined above, is a Banach space in which the functions are identified with almost every where equivalence. We should also mention that $L_2(\mathbb{T}) \subseteq L_1(\mathbb{T})$, which follows from the Cauchy-Schwarz inequality, and that the Lebesgue measure on $\mathbb{T}$ is finite.

Trigonometric Polynomials.

A *trigonometric polynomial* on $\mathbb{T}$ is a finite sum of the form

$$f(t) = \sum_{n=-N}^{N} a_n e^{int}, \quad t \in \mathbb{T},$$

where $a_n \in \mathbb{C}$. Notice that the trigonometric polynomials are bounded functions; consequently, they are in $L_p(\mathbb{T}),\ 1 \le p \le \infty$.

Convolutions.

If $f, g \in L_1(\mathbb{T})$, then the *convolution* of f and g, denoted by $f * g$, is defined by

$$f * g\,(\tau) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\tau - t) g(t)\ dt, \quad \tau \in \mathbb{T}.$$

The convolution operation in $L_1(\mathbb{T})$ is closed, commutative, associative and distributive (w.r.t. the addition).

Kernels.

A *summability kernel* on $\mathbb{T}$ is a sequence $\{P_N\}$ of continuous functions on $\mathbb{T}$ such that:

1) $\frac{1}{2\pi} \int\limits_{\mathbb{T}} P_N(t)\ dt = 1 \quad \forall\ N;$

2) $\|P_N\|_1 \leq \lambda \quad \forall\, N,$ for some constant λ;

3) for all $0 < \delta < \pi$,

$$\lim_{N\to\infty} \int_{\delta}^{2\pi-\delta} |P_N(t)|\, dt = 0.$$

If $f \in L_1(\mathbb{T})$, and k_n is a kernel, then $k_n * f \underset{n}{\longrightarrow} f$ in the L_1-norm.

As far as we are concerned in this thesis, the best servant and the most useful kernel is the *Fejér's kernel* $\{K_N\}_1^{\infty}$, which are trigonometric polynomials defined by

$$K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int}, \quad t \in \mathbb{T}.$$

An important property of the Fejér kernel is that $K_N \geq 0 \quad \forall\, N \geq 1$; and hence, $\|K_N\|_1 = 1 \quad \forall\, N \geq 1$.

Fourier Series on $\mathbb{T}$

For any $f \in L_1(\mathbb{T})$, the *Fourier coefficient* of f at $n \in \mathbb{Z}$ is defined by the formula

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int}\, dt.$$

Notice now that for $N \geq 1$, $0 < \hat{K}_N(n) = \left(1 - \frac{|n|}{N+1}\right) \leq 1$, $-N \leq n \leq N$, and $\hat{K}_N(n) = 0$ elsewhere.

The set of all functions $f \in L_p(\mathbb{T})$ such that $\hat{f}(n) = 0 \quad \forall\, n < 0$ is the *Hardy space* $H_p(\mathbb{T})$.

The *Fourier series* of $f \in L_1(\mathbb{T})$ is the trigonometric series $f(t) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$.

The *Riesz projections* ($\mathbb{P}_+$ and $\mathbb{P}_-$) of $f \in L_1(\mathbb{T})$ are defined by

$$(\mathbb{P}_+ f)(t) = \sum_{n>0} \hat{f}(n) e^{int}, \quad \text{and} \quad (\mathbb{P}_- f)(t) = \sum_{n<0} \hat{f}(n) e^{int}, \quad t \in \mathbb{T}.$$

Next we state some remarkable properties of Fourier coefficients.

Proposition 0.1. *Let* $f, g \in L_1(\mathbb{T})$, *and* $n \in \mathbb{Z}$, *then*

1) $(\widehat{f+g})(n) = \hat{f}(n) + \hat{g}(n)$.

2) For $\alpha \in \mathbb{C}$, $\quad (\widehat{\alpha f})(n) = \alpha \hat{f}(n)$.

3) If $\bar{f}(t) = \overline{f(t)}$, then $\hat{\bar{f}}\,(n) = \overline{\hat{f}(-n)}$. In particular, if $f \in L_1(\mathbb{T})$ is a real-valued function, then $\overline{\hat{f}(n)} = \hat{f}(-n)$.

4) If $m \in \mathbb{Z}$, and $h(t) = f(t)e^{imt}$, then $\hat{h}(n) = \hat{f}(n-m)$.

5) $(\widehat{f * g})(n) = \hat{f}(n)\,\hat{g}(n)$.

For a proof of the following theorems see [5, p. 29] or [12, p. 85].

Theorem 0.1. *If $f \in L_2(\mathbb{T})$, then the Fourier series of f converges to f in the L_2-norm.*

Theorem 0.2 (Riesz-Fischer). *If $\{c_n\}$ is a sequence of complex numbers such that*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty;$$

i.e. $\{c_n\} \in \ell_2(\mathbb{Z})$,¶ *then there exists an $f \in L_2(\mathbb{T})$ such that $\hat{f}(n) = c_n \quad \forall\, n \in \mathbb{Z}$.*

Theorem 0.3 (Parseval). *If $f, g \in L_2(\mathbb{T})$, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,\overline{g(t)}\,dt = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)};$$

in particular,

$$\|f\|_2 = \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\right)^{1/2} = \|\hat{f}\|_{\ell_2(\mathbb{Z})}.$$

Conventions:

• It is worth-noting that throughout this thesis (and for convenience), we are going to use $\|f\|_{\ell_2(\mathcal{J})}$ instead of $\|\hat{f}\|_{\ell_2(\mathcal{J})}$, whenever $\mathcal{J}$ is a set of integers; that is,

$$\|f\|_{\ell_2(\mathcal{J})} = \left(\sum_{n\in\mathcal{J}} |\hat{f}(n)|^2\right)^{1/2}.$$

¶Recall that $\{c_n\} \in \ell_2(\mathbb{Z}) \iff \|c_n\|_{\ell_2(\mathbb{Z})} = \left(\sum\limits_{n=-\infty}^{\infty} |c_n|^2\right)^{1/2} < \infty$.

- If n and m are integers, then we let $[n,m]$ to be the set of all integers k such that $n \le k \le m$. Similarly, $[n,m)$, $(n,m]$ and (n,m).
- For $f \in L_1(\mathbb{T})$, the spectrum of f, denoted by $\operatorname{spec}(f)$, is defined by

$$\operatorname{spec}(f) = \{n \in \mathbb{Z} \; : \hat{f}(n) \neq 0\}.$$

The next lemma contains some well-known facts about the spectrum of f. The fifth fact in this lemma is already stated in [8, p. 614], but without a proof; and since we are going to use it later to prove the main result of this thesis, we record its proof now.

Lemma 0.1. *Let $f, g \in L_2(\mathbb{T})$, then*

1) $\operatorname{spec}(\alpha f) = \operatorname{spec}(f), \quad \alpha \in \mathbb{C},\ \alpha \neq 0.$

2) $\operatorname{spec}(f+g) \subseteq \operatorname{spec}(f) \cup \operatorname{spec}(g).$

3) $\operatorname{spec}(f * g) = \operatorname{spec}(f) \cap \operatorname{spec}(g).$

4) $\operatorname{spec}(f\,g) \subseteq \operatorname{spec}(f) + \operatorname{spec}(g).$

5) If $\operatorname{spec}(f) \subseteq (-\infty, 0]$, *and* $\mathbf{Re} f \in L_\infty(\mathbb{T})$, *then* $\operatorname{spec}\left(e^f\right) \subseteq (-\infty, 0]$.

Proof. Remembering Proposition 0.1, (1), (2) and (3) become obvious.

(4) The strategy of the proof here is to find a sequence of functions $\{P_M\}$ such that

$$\widehat{P}_M(n) \underset{M}{\longrightarrow} (\widehat{fg})(n) \quad \forall\, n,$$

and $\operatorname{spec}(P_M) \subseteq \operatorname{spec}(f) + \operatorname{spec}(g) \quad \forall\ M$. So if $n \in \mathbb{Z}$, and $n \notin \operatorname{spec}(g) + \operatorname{spec}(f)$, then $n \notin \operatorname{spec}(P_M) \quad \forall\ M$. Consequently, $(\widehat{P}_M)(n) = 0 \quad \forall\ M$, which implies that $(\widehat{fg})(n) = 0$; i.e. $n \notin \operatorname{spec}(fg)$. We do that as follows.

First, by Proposition 0.1 (4), if $m \in \mathbb{Z}$, we have

$$\begin{aligned} n \in \operatorname{spec}(g(t)e^{imt}) &\iff \hat{g}(n-m) \neq 0 \\ &\iff n - m \in \operatorname{spec}(g) \\ &\iff n = n - m + m \in \operatorname{spec}(g) + \{m\}; \end{aligned}$$

hence, $\operatorname{spec}(g(t)e^{imt}) = \operatorname{spec}(g) + \{m\}$.

Next, let

$$S_M(t) = \sum_{m=-M}^{M} \hat{f}(m) e^{imt}, \quad M \geq 1,\ t \in \mathbb{T}.$$

Then, by the above result and (1) and (2) in this lemma, we get for $M \geq 1$,

$$\begin{aligned}
\mathrm{spec}(S_M\, g) &= \mathrm{spec}\left(\sum_{m=-M}^{M} g(t)\hat{f}(m) e^{imt} \right) \\
&\subseteq \bigcup_{m=-M}^{M} \mathrm{spec}\left(g(t)\hat{f}(m)e^{imt} \right) \\
&= \bigcup_{\substack{m=-M \\ m\in \mathrm{spec}(f)}}^{M} \mathrm{spec}\left(g(t)\hat{f}(m)e^{imt} \right) \\
&= \bigcup_{\substack{m=-M \\ m\in \mathrm{spec}(f)}}^{M} \mathrm{spec}\left(g(t)e^{imt} \right) \\
&= \bigcup_{\substack{m=-M \\ m\in \mathrm{spec}(f)}}^{M} \mathrm{spec}(g) + \{m\} \\
&= \mathrm{spec}(g) + \bigcup_{\substack{m=-M \\ m\in \mathrm{spec}(f)}}^{M} \{m\} \\
&\subseteq \mathrm{spec}(g) + \bigcup_{\substack{m=-\infty \\ m\in \mathrm{spec}(f)}}^{\infty} \{m\} \\
&= \mathrm{spec}(g) + \mathrm{spec}(f).
\end{aligned}$$

Now, by **Ineq. 4** (on page 16), the Cauchy-Schwarz inequality and Theorem 0.1,

$$\left| (\widehat{S_M\, g})(n) - (\widehat{fg})(n) \right| \leq \|S_M\, g - fg\|_1 \leq \|S_M - f\|_2\, \|g\|_2 \underset{M}{\longrightarrow} 0.$$

So, by letting $P_M = S_M\, g$, the proof of (4) is done.

(5) Consider $K_N * f,\ N \geq 1$, where K_N is the Fejér kernel of order N. First, note that

$$\begin{aligned}
\mathbf{Re}(K_N * f) &= \mathbf{Re}\left(K_N * (\mathbf{Re}f + i\mathbf{Im}f)\right) \\
&= \mathbf{Re}\left(K_N * \mathbf{Re}f + iK_N * \mathbf{Im}f\right) \\
&= K_N * \mathbf{Re}f,
\end{aligned}$$

because K_N are real-valued functions; in fact, they are ≥ 0.

Now, since $\mathbf{Re}f \in L_\infty(\mathbb{T})$, then for $N \geq 1$, and $t \in \mathbb{T}$,

$$\begin{aligned}
\mathbf{Re}(K_N * f)(\tau) &= K_N * \mathbf{Re}f\,(\tau) = \frac{1}{2\pi}\int_{\mathbb{T}} K_N(\tau - t)\mathbf{Re}f(t)\,dt \\
&\overset{(\star)}{\leq} \frac{1}{2\pi}\int_{\mathbb{T}} K_N(\tau - t)\|\mathbf{Re}f\|_\infty\,dt \\
&= \|\mathbf{Re}f\|_\infty < \infty,
\end{aligned}$$

where we have used in $(\star)$ the fact that $K_N \geq 0 \quad \forall\, N \geq 1$.

Consequently, since the exponential function is an increasing one,

$$\left|e^{K_N * f}\right| = e^{\mathbf{Re}(K_N * f)} = e^{K_N * \mathbf{Re}f} \leq e^{\|\mathbf{Re}f\|_\infty} < \infty \quad \forall\, N \geq 1. \tag{0.1}$$

Now since $f \in L_2(\mathbb{T}) \subseteq L_1(\mathbb{T})$, then $K_N * f \underset{N}{\longrightarrow} f$ in the L_1-norm; and hence there exists a subsequence $\{K_{N_\ell} * f\}_{\ell=1}^\infty$ such that

$$K_{N_\ell} * f\,(t) \underset{\ell}{\longrightarrow} f(t) \quad a.e\ (\mathbb{T}).$$

Thence,

$$e^{K_{N_\ell} * f\,(t)} \underset{\ell}{\longrightarrow} e^{f(t)} \quad a.e\ (\mathbb{T}).$$

Therefore, by (0.1) and the dominated convergence theorem, we obtain that

$$\left\|e^{K_{N_\ell} * f} - e^f\right\|_1 \underset{\ell}{\longrightarrow} 0.$$

But since for $n \in \mathbb{Z}$,

$$\left|\left(\widehat{e^{K_{N_\ell} * f}}\right)(n) - \left(\widehat{e^f}\right)(n)\right| \leq \left\|e^{K_{N_\ell} * f} - e^f\right\|_1,$$

then we have

$$\left(\widehat{e^{K_{N_\ell} * f}}\right)(n) \underset{\ell}{\longrightarrow} \left(\widehat{e^f}\right)(n) \quad \forall\, n \in \mathbb{Z}.$$

Following the same idea as in (4), we need now only to show that

$$\operatorname{spec}\left(e^{K_{N_\ell} * f}\right) \subseteq (-\infty, 0] \quad \forall\, \ell \geq 1.$$

To see that fix $N \geq 1$ and write $c_n = \hat{K}_N(n)\hat{f}(n)$, for convenience. Since $\operatorname{spec}(f) \subseteq (-\infty, 0]$, then

$$K_N * f\,(t) = \sum_{n=-N}^{0} c_n e^{int}, \quad t \in \mathbb{T}.$$

From (4) in this lemma, we have for all $N \geq 1$,

$$\begin{aligned}
\operatorname{spec}\left(e^{K_N * f(t)}\right) &= \operatorname{spec}\left(\prod_{n=-N}^{0} e^{c_n e^{int}}\right) \\
&\subseteq \sum_{n=-N}^{0} \operatorname{spec}\left(e^{c_n e^{int}}\right), \quad t \in \mathbb{T}. \qquad (0.2)
\end{aligned}$$

Now fix $n \in [-N, 0]$, and define for $M \geq 0$,

$$H_M(t) = \sum_{m=0}^{M} \frac{(c_n e^{int})^m}{m!} = \sum_{m=0}^{M} \frac{c_n^m e^{inmt}}{m!}, \quad t \in \mathbb{T}.$$

Then, by (1) and (2) in this lemma and bearing in mind that $n \in [-N, 0]$, we have for all $M \geq 0$,

$$\begin{aligned}
\operatorname{spec}(H_M) &\subseteq \bigcup_{m=0}^{M} \operatorname{spec}\left(\frac{c_n^m e^{inmt}}{m!}\right) \\
&\subseteq \bigcup_{m=0}^{M} \operatorname{spec}\left(e^{inmt}\right) \\
&= \bigcup_{m=0}^{M} \{nm\} \subseteq [nM, 0] \subseteq (-\infty, 0]. \qquad (0.3)
\end{aligned}$$

But since $0 \leq \hat{K}_N(n) \leq 1$, then $|c_n| \leq |\hat{f}(n)|$; hence

$$\begin{aligned}
\left|H_M(t) - e^{c_n e^{int}}\right| &= \left|\sum_{m=M+1}^{\infty} \frac{(c_n e^{int})^m}{m!}\right| \\
&\leq \sum_{m=M+1}^{\infty} \frac{|\hat{f}(n)|^m}{m!} \quad \forall\, t \in \mathbb{T}.
\end{aligned}$$

Therefore,

$$\left\|H_M(t) - e^{c_n e^{int}}\right\|_\infty \leq \sum_{m=M+1}^{\infty} \frac{|\hat{f}(n)|^m}{m!} \xrightarrow[M]{} 0,$$

which yields that for all $k \in \mathbb{Z}$,

$$\begin{aligned}
\left|\widehat{H}_M(k) - \widehat{\left(e^{c_n e^{int}}\right)}(k)\right| &\leq \left\|H_M(t) - e^{c_n e^{int}}\right\|_1 \\
&\leq \left\|H_M(t) - e^{c_n e^{int}}\right\|_\infty \xrightarrow[M]{} 0.
\end{aligned}$$

Using the strategy in (4), we get the following. If $k \notin (-\infty, 0]$, then (0.3) yields that $k \notin \operatorname{spec}(H_M) \;\; \forall\; M \geq 0$; and hence, by the last result, $k \notin \operatorname{spec}\left(e^{c_n e^{int}}\right)$. Therefore, $\operatorname{spec}\left(e^{c_n e^{int}}\right) \subseteq (-\infty, 0]$.

Since $n \in [-N, 0]$ is arbitrary, then $\operatorname{spec}\left(e^{c_n e^{int}}\right) \subseteq (-\infty, 0] \quad \forall\ n \in [-N, 0]$. Substitute back in (0.2), we obtain

$$\operatorname{spec}\left(e^{K_N * f}\right) \subseteq (-\infty, 0] \quad \forall\ N \geq 1.$$

By this we have proved more than enough. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Besov Spaces

Next we turn our attention to the definition of the Besov spaces, which play an essential role in reformulating the main result of this thesis. The following description of the Besov spaces is mentioned in the introduction of [10].*

The definition involves convolutions with special kernels W_n, $n \in \mathbb{Z}$; and so we need first to describe them. For $n > 0$, W_n is defined to be a trigonometric polynomial such that $\widehat{W}_n$ is a linear function on the intervals $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$, $\widehat{W}_n(2^n) = 1$, and $\widehat{W}_n \equiv 0$ outside $(2^{n-1}, 2^{n+1})$. From that, $W_n = \overline{W}_{-n}$ for all $n < 0$; and finally, $W_0(t) = e^{-it} + 1 + e^{it}$, $t \in \mathbb{T}$.

To be more precise, take for $n > 0$,

$$\widehat{W}_n(k) = \begin{cases} \hat{K}_{2^{n-1}-1}(2^n - k) = \frac{k-2^{n-1}}{2^{n-1}}; & k \in [2^{n-1}, 2^n] \\ \hat{K}_{2^n-1}(2^n - k) \ = \frac{2^{n+1}-k}{2^n}; & k \in [2^n, 2^{n+1}] \\ 0; & \text{otherwise} \end{cases}, \tag{0.4}$$

let $W_n = \overline{W}_{-n}$ for $n < 0$, and $W_0(t) = e^{-it} + 1 + e^{it}$, $t \in \mathbb{T}$.

Then, we record the following properties of W_n.

Proposition 0.2.

1) $0 \leq \widehat{W}_n(k) \leq 1, \quad n, k \in \mathbb{Z}$.

2) $\widehat{W}_n(k) = \overline{\widehat{W}_n(k)} = \widehat{W}_{-n}(-k), \quad n, k > 0$.

3) $\widehat{W}_n(k) + \widehat{W}_{n-1}(k) = 1, \quad n > 0$, *and* $k \in [2^{n-1}, 2^n]$.

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*For other equivalent definitions and abstract results about the Besov spaces, see J. Peetre, *New thoughts on Besov spaces*, (Duke Univ. Math. Ser. **1**), Duke Univ., Durham, N.C., 1976.

4) $\widehat{W}_n(k) + \widehat{W}_{n-1}(k) = 1, \quad n < 0,$ *and* $k \in [-2^{-n}, -2^{-n-1}]$.

5) For $n > 0$, *and* $k \in [2^{n-1}, 2^n]$,

$$\left[\widehat{W}_n(k)\right]^2 + \left[\widehat{W}_{n-1}(k)\right]^2 \geq \frac{1}{2}.$$

***Proof*.** The first four properties are easy to prove.
(5) Fix $n > 0$ and $k \in [2^{n-1}, 2^n]$. Let $a = \widehat{W}_n(k)$ and $b = \widehat{W}_{n-1}(k)$. Then ,by (3), $a + b = 1$. If one of them is equal to zero or $a = b = \frac{1}{2}$, then we are done; so we may assume that $1 > a > \frac{1}{2} > b > 0$. Now, since $a + b = \frac{1}{2} + \frac{1}{2}$, then $a - \frac{1}{2} = \frac{1}{2} - b$. Since $a > b$, then $a(a - \frac{1}{2}) > b(\frac{1}{2} - b)$; hence $a^2 - \frac{1}{2}a > \frac{1}{2}b - b^2$; and therefore, $a^2 + b^2 > \frac{1}{2}(a + b) = \frac{1}{2}$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Definition 0.1. For $p \geq 1$, $q < \infty$ and $s \in \mathbb{R}$, the *Besov space* $B^s_{p\,q}$ is defined to be the set of all functions $f \in L_p(\mathbb{T})$ such that

$$\|f\|_{B^s_{p\,q}} := \left(\sum_{n \in \mathbb{Z}} \left(2^{|n|s} \|W_n * f\|_p \right)^q \right)^{1/q} < \infty.$$

It should be noted here that in the last chapter of this thesis, where we make use of the Besov space $B^{-1/2}_{2\,1}$, we will be referring to this important section.

Useful Inequalities

Although we have already used some of them in this chapter, we find it more helpful to state some of the well-known inequalities that we are going to use later in the proof of many results in this thesis. The first four inequalities are proved; and references, where proofs can be found, are given for the others.

Ineq. 1. *If* a *and* b *are non-negative real numbers, then*

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \leq \sqrt{2}\,\sqrt{a+b}.$$

***Proof*.** Just notice that for $a, b \geq 0$,

$$2\sqrt{ab} \leq (\sqrt{a} - \sqrt{b})^2 + 2\sqrt{ab} = a + b,$$

consequently,

$$a + b \leq (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} \leq 2(a + b).$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Ineq. 2. *Let x be any real number such that $0 \leq x \leq \frac{1}{2}$, then*

$$\ln(1 - x) \geq -2x.$$

***Proof*.** This is easy. In fact, if $0 \leq x \leq \frac{1}{2}$, then

$$\begin{aligned} -\ln(1-x) &= \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} \sum_{k=0}^{\infty} \frac{2x^k}{k+2} \\ &\leq x + \frac{x}{2} \sum_{k=0}^{\infty} x^k \\ &= x + \frac{x}{2} \cdot \frac{1}{1-x} \\ &\leq x + \frac{x}{2} \cdot 2 = 2x. \end{aligned}$$

Another proof can be given by using the mean value theorem. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Ineq. 3. *For all $z \in \mathbb{C}$ with* $\mathbf{Re} z \geq 0$,

$$|e^{-z} - 1| \leq |z|.$$

***Proof*.** First the inequality is trivial if $z = 0$; so assume that $z \neq 0$. Consider the function $f(z) = \frac{e^{-z}-1}{z}$, $z \neq 0$. Note that $f(z)$ is bounded on $\mathbf{Re} z \geq 0$. Indeed,

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^k z^{k-1}}{k!}, \quad \text{and} \quad \lim_{|z| \to \infty} f(z) = 0.$$

Hence, the maximum modulus theorem applies on $\mathbf{Re} z \geq 0$. But if $z = iy$, then

$$|e^{iy} - 1| = |e^{iy/2} - e^{-iy/2}| = 2|\sin(y/2)| \leq 2|y/2| = |y| = |iy|.$$

Therefore, on $\mathbf{Re} z \geq 0$,

$$\frac{|e^z - 1|}{|z|} \leq \sup_{y \in \mathbb{R}} \frac{|e^{iy} - 1|}{|iy|} \leq 1.$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Ineq. 4. *If $f \in L_1(\mathbb{T})$, then for all $n \in \mathbb{Z}$,*

$$|\hat{f}(n)| \leq \|f\|_1,$$

moreover, if f is also bounded, then

$$|\hat{f}(n)| \leq \|f\|_1 \leq \|f\|_\infty.$$

***Proof*.** Just consider this

$$\begin{aligned} |\hat{f}(n)| &\leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|\, |e^{int}|\, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|\, dt = \|f\|_1 \\ &\leq \frac{1}{2\pi} \int_{\mathbb{T}} \|f\|_\infty\, dt = \|f\|_\infty. \end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Ineq. 5 (Cauchy-Schwarz). *If $f, g \in L_2(\mathbb{T})$, then*

$$\|f\, g\|_1 \leq \|f\|_2\, \|g\|_2.$$

***Proof*.** See [12, p. 63].

Ineq. 6. *If $f, g \in L_1(\mathbb{T})$,*

$$\|f * g\|_1 \leq \|f\|_1\, \|g\|_1.$$

***Proof*.** See [5, p. 4].

Ineq. 7 (Young). *If $f \in L_1(\mathbb{T})$ and $g \in L_p(\mathbb{T})$, $1 \leq p \leq \infty$, then*

$$\|f * g\|_p \leq \|f\|_1\, \|g\|_p.$$

***Proof*.** See [1, p. 232].

Chapter 1

Generalizations of Hardy's Inequality

An inequality of G. H. Hardy and J. E. Littlewood ([4], 1948) states that if $1 < p \leq 2$, $f \in L_p(\mathbb{T})$, and if

$$|\tilde{\hat{f}}(0)|, |\tilde{\hat{f}}(1)|, \ |\tilde{\hat{f}}(-1)|, \ |\tilde{\hat{f}}(2)|, \ |\tilde{\hat{f}}(-2)|, \ \cdots$$

is the sequence of $|\hat{f}(n)|$ arranged in descending order of magnitude, then there is a constant $a_p > 0$ depending on p such that

$$\left(\sum_{n \in \mathbb{Z}} \frac{|\tilde{\hat{f}}(n)|}{(|n|+1)^{p-2}} \right)^{1/p} \leq a_p \, \|f\|_p.$$

It is reported also in the same paper that the inequality is not true for $p = 1$. However, as we mentioned in the introduction, Hardy's inequality ([3], 1927) states that there is a constant $c > 0$ such that for all $f \in H_1(\mathbb{T}) = \{f \in L_1(\mathbb{T}) \ : \ \mathbb{P}_- f = 0\}$,

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \leq c \|f\|_1; \tag{1.1}$$

and it is well-known that this inequality is not valid for all $f \in L_1(\mathbb{T})$.

This chapter is devoted to expose briefly two well-known generalizations of Hardy's inequality that are "related" to the mixed-norm generalization (Theorem 2.1, p. 23). The two generalizations were actually stated in terms of finite complex Borel measures on $\mathbb{T}$, but in order to serve the thesis' subject, we "restate" them for functions $f \in L_1(\mathbb{T})$.

The generalized Hardy's inequality of McGehee, Pigno and Smith

The truth of the well-known Littlewood conjecture, concerning the L_1-norm of exponential sums, was established in 1981 by the team of O. C. McGehee, L. Pigno and B. Smith [8] as a special case of the following generalization of Hardy's inequality*.

Theorem 1.1. *There is a real number* $c > 0$ *such that given any set of integers* $S = \{n_1, n_2, \cdots\} \subseteq \mathbb{Z}$ *and* $f \in L_1(\mathbb{T})$ *such that* $n_1 < n_2 < \cdots$, *and* $\operatorname{spec}(f) \subseteq S$, *then*

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{k} \leq c\,\|f\|_1.$$

The proof of the theorem was remarkably simple and was based on a clever construction of bounded functions whose Fourier coefficients have "desired" properties. To clarify the idea, we explain here how the authors formed the construction.

Let $\{S_j\}_{j=1}^{\infty}$ be the partition of S such that $\operatorname{card}(S_j) = 4^j$ and $S_0 = \{n_1\}$, $S_1 = \{n_2, n_3, n_4, n_5\}$, $S_3 = \{n_6, \cdots, n_{21}\}$, and so on. Define trigonometric polynomials f_j, $j = 0, 1, 2, \cdots$, such that

1) $\hat{f}_j(n) = 0$, if $n \notin S_j$;
2) $|\hat{f}_j(n)| = 4^{-j}$ if $n \in S_j$;
3) $\hat{f}_j\,\hat{f} \geq 0$.

Write $|f_j| = \sum\limits_{-\infty}^{\infty} a_n e^{int}$, and define for $j = 0, 1, 2, \cdots$,

$$h_j(t) = \frac{1}{4}\left(a_0 + 2\sum_{-\infty}^{-1} a_n e^{int}\right), \quad t \in \mathbb{T}.$$

---

*The Littlewood conjecture was also proved independently by S. V. Konjagin (S. V. Konjagin, *On a problem of Littlewood*, Math. U.S.S.R. Izvestia Vol. **18** (2) (1982), 205-225 [translation of Izvestia Akad. Nauk U.S.S.R., Ser. Mat., **45** (2) (1981), 243-265]). However, according to [2], it seems that Konjagin's proof does not contain any generalization of Hardy's inequality.

Then the construction was defined inductively as follows:

$$\begin{aligned} F_0 &= \frac{1}{5} f_0; \\ F_{j+1} &= F_j e^{-h_{j+1}} + \frac{1}{5} f_{j+1}, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{1.2}$$

After that the authors pursued and showed that the above construction satisfies the following crucial features:

1) $\|F_j\|_\infty \leq 1, \quad j \geq 0.$

2) If $n \in S_j,\ 0 \leq j \leq m,$ then

$$\left| \hat{F}_m(n) - \frac{1}{5} \hat{f}_j(n) \right| \leq \frac{1}{10} |\hat{f}_j(n)|.$$

3) If $n_k \in S_j,\ 0 \leq j \leq m,$ then

$$\mathbf{Re} \left(\hat{F}_m \, \hat{f} \right) (n_k) \geq \frac{|\hat{f}(n_k)|}{30k}.$$

In 1983, J. J. F. Fournier [2] reported three different constructions that play the same role and satisfy the same properties as the construction (1.2). These constructions are as follows

1) It should be mentioned that this construction was actually established by L. Pigno and B. Smith in [11] for a related purpose. Let $0 < a < \frac{1}{2}$, and define inductively

$$\begin{aligned} F_0 &= 1; \\ F_{j+1} &= a f_{j+1} + \left(1 - 4a^2 |f_{j+1}|^2\right) F_j - a \bar{f}_{j+1} F_j^2, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{1.3}$$

2) Let $0 < a < 1$, and define the construction by letting

$$\begin{aligned} F_0 &= 0; \\ F_{j+1} &= \frac{F_j + a f_{j+1}}{1 + a \bar{f}_{j+1} F_j}, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{1.4}$$

3) Let $0 < a < 1$. Since the function $\log(1 - a|f_{j+1}|)$ is integrable, then there is an outer function¶ G_{j+1} such that $|G_{j+1}| = 1 - a|f_{j+1}|$ and

$$\hat{G}_{j+1}(0) = \exp \left(\frac{1}{2\pi} \int_{\mathbb{T}} \log(1 - a|f_{j+1}|) \, dt \right).$$

¶For a definition of outer function see [12, p. 342].

Then define the construction as follows:

$$\begin{aligned} F_0 &= 0; \\ F_{j+1} &= \bar{G}_{j+1} F_j + a f_{j+1}, \quad j = 0, 1, 2, \cdots. \end{aligned} \tag{1.5}$$

The mixed-norm generalization of Hardy's inequality (Theorem 2.1, p. 23), as we mentioned in the abstract, was already proved by I. Klemes [6] in 1993. The proof depends mainly on building a certain trigonometric polynomial F that satisfies the following:

1) $F \in L_\infty(\mathbb{T})$.

2) There is a constant $c > 0$ such that

$$\left(\sum_{4^{j-1} \le n < 4^j} |\hat{F}(-n)|^2 \right)^{1/2} \le c\, 4^{-j/2}, \quad j \ge 1.$$

3) If f_j, $j = 1, 2, \cdots$, are defined by either $f_j = 0$, if $\hat{f}(n) = 0 \quad \forall\, n \in [4^{j-1}, 4^j)$, or otherwise by

$$f_j(t) = 4^{-j/2} \left(\sum_{4^{j-1} \le k < 4^j} |\hat{f}(k)|^2 \right)^{-1/2} \sum_{4^{j-1} \le n < 4^j} \hat{f}(n) e^{int}, \quad t \in \mathbb{T},$$

then

$$\left(\sum_{4^{j-1} \le n < 4^j} |\hat{F}(n) - \hat{f}_j(n)|^2 \right)^{1/2} \le \frac{1}{2} 4^{-j/2}, \quad j \ge 1.$$

The trigonometric polynomial F was formed via using the construction (1.3); however, I. Klemes remarked in the same paper that the other three constructions, we mentioned above, and "with some modifications (such as convolving with Fejér kernel at each step), each of them would also work here".

It is well-known (as noted in [6]) that the mixed-norm generalization is much stronger than Hardy's inequality if $f \in H_1(\mathbb{T})$, because

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} \le b \sum_{j=1}^{\infty} \left(4^{-j} \sum_{4^{j-1} \le n < 4^j} |\hat{f}(n)|^2 \right)^{1/2},$$

for some constant $b > 0$. In fact, via the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{4^{j-1}\le n<4^j} \frac{|\hat{f}(n)|}{n} &\le \left(\sum_{4^{j-1}\le n<4^j} |\hat{f}(n)|^2\right)^{1/2} \cdot \left(\sum_{4^{j-1}\le n<4^j} \frac{1}{n^2}\right)^{1/2} \\ &\le \left(\sum_{4^{j-1}\le n<4^j} |\hat{f}(n)|^2\right)^{1/2} \cdot \left(\sum_{4^{j-1}\le n<4^j} \frac{1}{4^{2(j-1)}}\right)^{1/2} \\ &= \left(\sum_{4^{j-1}\le n<4^j} |\hat{f}(n)|^2\right)^{1/2} \cdot \left(\frac{3\cdot 4^j}{4^{2(j-1)}}\right)^{1/2} \\ &= 4\sqrt{3}\left(4^{-j}\sum_{4^{j-1}\le n<4^j} |\hat{f}(n)|^2\right)^{1/2};\end{aligned}$$

hence,

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|}{n} = \sum_{j=1}^{\infty}\left(\sum_{4^{j-1}\le n<4^j} \frac{|\hat{f}(n)|}{n}\right) \le 4\sqrt{3}\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\le n<4^j} |\hat{f}(n)|^2\right)^{1/2}.$$

This fact can also be infered from [9, p. 223].

Hardy's inequality and the Besov spaces

In [10, §3.6], V. V. Peller and S. V. Khrushchev treated the reconstruction problem in the space of all finite complex Borel measures on $\mathbb{T}$. The problem consists in finding conditions on X such that if μ is a finite complex Borel measure and $\mathbb{P}_-\mu \in X$, then $\mu \in X$. In this section, we are going to display another well-known generalization of Hardy's inequality, which is stated in this paper as a corollary of a theorem on the reconstruction problem. Again for the sake of the thesis' subject, we are going to expose the results in terms of functions $f \in L_1(\mathbb{T})$.

Theorem 1.2 (Peller-Khrushchev). *Let $1 < p < \infty$ and $f \in L_1(\mathbb{T})$. Then*

$$\mathbb{P}_- f \in B_{pp}^{-1/p'} \Longrightarrow f \in B_{pp}^{-1/p'}, \tag{1.6}$$

where $p' = \frac{p}{p-1}$.

Applying the definition of the Besov space $B_{pp}^{-1/p'}$ for $p = 2$, the theorem gives the following generalization of Hardy's inequality.

Corollary 1.1. *If $f \in L_1(\mathbb{T})$, then*

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|^2}{n} < \infty \Longrightarrow \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n} < \infty.$$

In addition, it is remarked in the same paper that if $s \in \mathbb{Z}$, then

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|^{2s}}{n} < \infty \Longrightarrow \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^{2s}}{n} < \infty. \tag{1.7}$$

It is an open problem to find numbers $s \geq \frac{1}{2}$ for which (1.7) holds. For $s = \frac{1}{2}$, (1.7) becomes (0.1).

P. Koosis (jointly with S. Picorides) in [7] gave a new simple and elegant proof of Corollary 1.1; in fact a stronger result.

Theorem 1.3 (Koosis). *If $f \in L_1(\mathbb{T})$, and $\sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n} < \infty$, then*

$$\left| \sum_{n=1}^{\infty} \frac{|\hat{f}(n)|^2}{n} - \sum_{n=1}^{\infty} \frac{|\hat{f}(-n)|^2}{n} \right| \leq \pi \|f\|_1^2.$$

It should be noted here that we will be referring to this crucial section at the end of chapter 2.

Chapter 2

Mixed-Norm Generalization of Hardy's Inequality

The objects of this thesis are achieved here. The mixed-norm generalization of Hardy's inequality, which is the main result, is proved first. In the next section, we prove that the main result can be expressed using the norm of the Besov space $B_{2\,1}^{-1/2}$.

Main Result

Theorem 2.1 (Klemes). *There is an absolute constant* $c > 0$ *such that for all functions* $f \in L_1(\mathbb{T})$,

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(n)|^{2}\right)^{1/2}\leq c\|f\|_{1}+c\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^{j}}|\hat{f}(-n)|^{2}\right)^{1/2}. \tag{2.1}$$

Proof. First, it is easy to see that if the theorem is true for all trigonometric polynomials in $L_1(\mathbb{T})$, then it is also true for all functions $f \in L_1(\mathbb{T})$. Indeed, suppose the inequality (2.1) is true for all trigonometric polynomials in $L_1(\mathbb{T})$ with some absolute constant $c > 0$ and let $f \in L_1(\mathbb{T})$ be any arbitrary function.

Consider $K_N * f, \quad N \geq 1$, where K_N is the Fejér kernel of order N. Since for all $N \geq 1$,

$$K_N * f\ (t) = \sum_{n=-N}^{N} \hat{K}_N(n)\, \hat{f}(n)\, e^{int},$$

and

$$\|K_N * f\|_1 \leq \|K_N\|_1 \|f\|_1 = \|f\|_1 < \infty,$$

then $K_N * f$ is a trigonometric polynomial in $L_1(\mathbb{T}) \quad \forall\, N \geq 1$.

Therefore, $K_N * f$ satisfies the inequality (2.1) $\forall\, N \geq 1$; i.e.

$$\sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} \left[\hat{K}_N(n)\right]^2 |\hat{f}(n)|^2 \right)^{1/2} \leq c\|K_N * f\|_1 + c \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} \left[\hat{K}_N(-n)\right]^2 |\hat{f}(-n)|^2 \right)^{1/2},$$

where $I_j = [4^{j-1}, 4^j)$.

Now, recall that $0 \leq \hat{K}_N(n) \leq 1 \quad \forall\, N \geq 1$, and $\forall\, n \in \mathbb{Z}$. Hence, for all $M, N \geq 1$, we have

$$\begin{aligned} R_M(N) &:= \sum_{j=1}^{M} \left(4^{-j} \sum_{n \in I_j} \left[\hat{K}_N(n)\right]^2 |\hat{f}(n)|^2 \right)^{1/2} \\ &\leq \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} \left[\hat{K}_N(n)\right]^2 |\hat{f}(n)|^2 \right)^{1/2} \\ &\leq c\|K_N * f\|_1 + c \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} \left[\hat{K}_N(-n)\right]^2 |\hat{f}(-n)|^2 \right)^{1/2} \\ &\leq c\|f\|_1 + c \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} |\hat{f}(-n)|^2 \right)^{1/2}. \end{aligned}$$

Since $R_M(N)$ is a finite sum for all $N \geq 1$ and $\left[\hat{K}_N(n)\right]^2 \underset{N}{\longrightarrow} 1 \quad \forall\, n \in \mathbb{N}$, then $\forall\, M \geq 1$,

$$\lim_{N \to \infty} R_M(N) = \sum_{j=1}^{M} \left(4^{-j} \sum_{n \in I_j} |\hat{f}(n)|^2 \right)^{1/2} \leq c\|f\|_1 + c \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in I_j} |\hat{f}(-n)|^2 \right)^{1/2}.$$

Hence, our claim now follows directly because

$$\begin{aligned}\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n\in I_j}|\hat{f}(n)|^2\right)^{1/2} &= \lim_{M\to\infty}\lim_{N\to\infty} R_M(N)\\ &\leq c\|f\|_1 + c\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n\in I_j}|\hat{f}(-n)|^2\right)^{1/2}.\end{aligned}$$

Having proved that, we can proceed now to complete the proof of the theorem, assuming that $f \in L_1(\mathbb{T})$ is a trigonometric polynomial. Say

$$f(t) = \sum_{n=-4^J+1}^{4^J-1} \hat{f}(n)e^{int}, \qquad t \in \mathbb{T},$$

where J is the smallest non-negative integer such that $\operatorname{spec}(f) \subseteq (-4^J, 4^J)$. Since $0 \leq c\|f\|_1$ is true for $c > 0$, then there is nothing to say when $J = 0$. Hence, we can assume also that $J \geq 1$.

For convenience, write $a_n = \hat{f}(n), \quad n \in \mathbb{Z}$.

Define trigonometric polynomials $\varphi_j, \quad j = 1, \cdots, J$ *, by

$$\varphi_j(t) = \begin{cases} 0\,; & \text{if } a_n = 0 \quad \forall\, n \in I_j \\ \\ 4^{-j/2}\left(\sum\limits_{k\in I_j}|a_k|^2\right)^{-1/2}\sum\limits_{n\in I_j} a_n e^{int}\,; & \text{otherwise} \end{cases}, \quad t \in \mathbb{T}, \tag{2.2}$$

where I_j still denotes the interval $[4^{j-1}, 4^j)$.

For the time being, assume that the following proposition and theorem are true.

Proposition 2.1. *For* $n \in \mathbb{N}$, *define*

$$B_n = \lim_{L\to\infty}\prod_{\ell=j}^{L}\hat{K}_{(\ell)}(n),$$

where j *is the unique integer such that* $n \in I_j = [4^{j-1}, 4^j)$, *and* $K_{(\ell)}$, *from now and on, denotes* $K_{2\cdot 4^\ell - 3}$, *the Fejér kernel of order equal to* $2 \cdot 4^\ell - 3$. *Then* B_n *is well-defined and* $B_n > 1/4$.

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*In this chapter, j is just an index which may have different limits in different occurrences.

Theorem 2.2. *There exist absolute constants $c_1,\ c_2 > 0$ and a trigonometric polynomial F with the following properties:*

(1) $\|F\|_\infty \le c_1$;

(2) $\left(\sum_{n\in I_j} |\hat{F}(-n)|^2\right)^{1/2} \le c_2 4^{-j/2}, \qquad j \ge 1;$

(3) for $1 \le j \le J, \qquad \left(\sum_{n\in I_j} |\hat{F}(n) - b_n\hat{\varphi}_j(n)|^2\right)^{1/2} \le \frac{1}{8}4^{-j/2},$

where $b_n = \prod_{\ell=j}^{J} \hat{K}_{(\ell)}(n), \quad n \in I_j = [4^{j-1}, 4^j).$

Then the proof of the main theorem continues by using a standard duality argument as follows.

Consider the function F obtained from Theorem 2.2. Since $\|F\|_\infty \le c_1$, then we have

$$\begin{aligned} c_1\|f\|_1 &\ge \|F\|_\infty \int_{\mathbb{T}} |f(t)|\ \frac{dt}{2\pi} \\ &\ge \int_{\mathbb{T}} \left|\overline{F(t)}\right| |f(t)|\ \frac{dt}{2\pi} \\ &\ge \left|\int_{\mathbb{T}} \overline{F(t)} f(t)\ \frac{dt}{2\pi}\right|. \end{aligned} \tag{2.3}$$

Notice that since f and F are trigonometric polynomials, and so in $L_2(\mathbb{T})$, then, by Parseval's theorem, we have

$$\int_{\mathbb{T}} \overline{F(t)} f(t)\ \frac{dt}{2\pi} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{F}(n)}.$$

Recalling that $a_n = \hat{f}(n)$ and substituting in (2.3), we get

$$\begin{aligned} c_1\|f\|_1 &\ge \left|a_0\overline{\hat{F}(0)} + \sum_{n>0} a_n\overline{\hat{F}(n)} + \sum_{n<0} a_n\overline{\hat{F}(n)}\right| \\ &\ge \left|\sum_{n>0} a_n\overline{\hat{F}(n)}\right| - \left|a_0\overline{\hat{F}(0)} + \sum_{n<0} a_n\overline{\hat{F}(n)}\right| \\ &\ge \left|\sum_{n>0} a_n\overline{\hat{F}(n)}\right| - \left(\left|a_0\overline{\hat{F}(0)}\right| + \left|\sum_{n<0} a_n\overline{\hat{F}(n)}\right|\right), \end{aligned} \tag{2.4}$$

where we have used the triangle inequality in the last two steps.

Now notice that $\left|\overline{\hat{F}(0)}\right| = \left|\hat{F}(0)\right| \leq \|F\|_\infty \leq c_1$, and $|a_0| = |\hat{f}(0)| \leq \|f\|_1$.

Hence, rewriting (2.4), by using the above information, we get

$$\begin{aligned}
\left|\sum_{n>0} a_n \overline{\hat{F}(n)}\right| &\leq c_1\|f\|_1 + \left|a_0\overline{\hat{F}(0)}\right| + \left|\sum_{n<0} a_n \overline{\hat{F}(n)}\right| \\
&\leq c_1\|f\|_1 + c_1\|f\|_1 + \sum_{n<0} |a_n| \left|\overline{\hat{F}(n)}\right| \\
&= 2c_1\|f\|_1 + \sum_{j=1}^{\infty}\sum_{n\in I_j} |a_{-n}||\hat{F}(-n)| \\
&\leq 2c_1\|f\|_1 + \sum_{j=1}^{\infty}\left[\left(\sum_{n\in I_j}|a_{-n}|^2\right)^{1/2}\left(\sum_{n\in I_j}|\hat{F}(-n)|^2\right)^{1/2}\right] \\
&\leq 2c_1\|f\|_1 + c_2\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n\in I_j}|a_{-n}|^2\right)^{1/2}, \qquad (2.5)
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and Theorem 2.2 (2) in the last two lines, respectively.

On the left-hand side of (2.5), we have

$$\begin{aligned}
\left|\sum_{n>0} a_n \overline{\hat{F}(n)}\right| &= \left|\sum_{j=1}^{\infty}\left(\sum_{n\in I_j} a_n \overline{\hat{F}(n)}\right)\right| \\
&\geq \sum_{j=1}^{\infty}\mathbf{Re}\left(\sum_{n\in I_j} a_n \overline{\hat{F}(n)}\right). \qquad (2.6)
\end{aligned}$$

Now fix $j \geq 1$. If $a_n = 0 \quad \forall\, n \in I_j$, then

$$\mathbf{Re}\left(\sum_{n\in I_j} a_n \overline{\hat{F}(n)}\right) = 0 = \left(4^{-j}\sum_{n\in I_j}|a_n|^2\right)^{1/2}. \qquad (2.7)$$

If $a_n \neq 0$ for some $n \in I_j$, then we next prove that

$$\mathbf{Re}\left(\sum_{n\in I_j} \hat{\varphi}_j(n)\overline{\hat{F}(n)}\right) \geq \frac{1}{8}4^{-j}.$$

First, since for $n \in I_j, \quad 0 < \hat{K}_\ell(n) < 1 \quad \forall\, \ell \geq j$, then, by using Proposition 2.1, we obtain that

$$b_n = \prod_{\ell=j}^{J} \hat{K}_{(\ell)}(n) > B_n = \lim_{L\to\infty} \prod_{\ell=j}^{L} \hat{K}_{(\ell)}(n) > \frac{1}{4}. \tag{2.8}$$

Also, notice that

$$\|\varphi_j\|_2 = \|\varphi_j\|_{\ell_2(I_j)} = 4^{-j/2} \left(\sum_{k\in I_j} |a_k|^2\right)^{-1/2} \left(\sum_{n\in I_j} |a_n|^2\right)^{1/2} = 4^{-j/2}. \tag{2.9}$$

Hence, using (2.8) and (2.9), we get

$$\sum_{n\in I_j} b_n \hat{\varphi}_j(n)\overline{\hat{\varphi}_j(n)} = \sum_{n\in I_j} b_n |\hat{\varphi}_j(n)|^2 > \frac{1}{4}\|\varphi_j\|_2^2 = \frac{1}{4}4^{-j}. \tag{2.10}$$

Therefore, we obtain

$$\begin{aligned}
\mathbf{Re}\left(\sum_{n\in I_j} \hat{\varphi}_j(n)\overline{\hat{F}(n)}\right) &= \mathbf{Re}\left(\sum_{n\in I_j} \hat{\varphi}_j(n)\left(\overline{b_n\hat{\varphi}_j(n) + \hat{F}(n) - b_n\hat{\varphi}_j(n)}\right)\right)\\
&= \mathbf{Re}\left(\sum_{n\in I_j} b_n\hat{\varphi}_j(n)\overline{\hat{\varphi}_j(n)}\right) + \mathbf{Re}\left(\sum_{n\in I_j}\hat{\varphi}_j(n)\left(\overline{\hat{F}(n) - b_n\hat{\varphi}_j(n)}\right)\right)\\
&> \frac{1}{4}4^{-j} - \sum_{n\in I_j} |\hat{\varphi}_j(n)| \left|\overline{\hat{F}(n) - b_n\hat{\varphi}_j(n)}\right|\\
&> \frac{1}{4}4^{-j} - \|\varphi_j\|_{\ell_2(I_j)} \left(\sum_{n\in I_j} |\hat{F}(n) - b_n\hat{\varphi}_j(n)|^2\right)^{1/2},
\end{aligned}$$

where we have used (2.10) and the Cauchy-Schwarz inequality in the last two lines, respectively. Using Theorem 2.2 (3) and (2.9), the last estimate becomes

$$\mathbf{Re}\left(\sum_{n\in I_j} \hat{\varphi}_j(n)\overline{\hat{F}(n)}\right) > \frac{1}{4}4^{-j} - 4^{-j/2}\cdot\frac{1}{8}4^{-j/2} = \frac{1}{8}4^{-j}.$$

Now, since

$$a_n = 4^{j/2}\left(\sum_{k\in I_j} |a_k|^2\right)^{1/2} \hat{\varphi}_j(n),$$

then via using the last estimate

$$\begin{aligned}\mathbf{Re}\left(\sum_{n\in I_j} a_n\overline{\hat{F}(n)}\right) &= 4^{j/2}\left(\sum_{k\in I_j}|a_k|^2\right)^{1/2}\mathbf{Re}\left(\sum_{n\in I_j}\hat{\varphi}_j(n)\overline{\hat{F}(n)}\right)\\ &> \frac{1}{8}\left(4^{-j}\sum_{k\in I_j}|a_k|^2\right)^{1/2}. \end{aligned} \tag{2.11}$$

From (2.6), (2.7) and (2.11), we get

$$\left|\sum_{n>0} a_n\overline{\hat{F}(n)}\right| \geq \sum_{j=1}^{\infty}\frac{1}{8}\left(4^{-j}\sum_{n\in I_j}|a_n|^2\right)^{1/2}. \tag{2.12}$$

Finally, collecting estimates (2.5) and (2.12), and recalling that $a_n = \hat{f}(n)$, we have

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n\in I_j}|\hat{f}(n)|^2\right)^{1/2} \leq 16c_1\|f\|_1 + 8c_2\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq n<4^j}|\hat{f}(-n)|^2\right)^{1/2}.$$

Letting $c = 8\max(2c_1, c_2) = 1024$, the proof is done. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Now, we need only to prove the above Proposition 2.1 and Theorem 2.2. Before we do that, it is very useful to compare the above proof and the original one given by I. Klemes in [6]. Both proofs have used the standard duality argument after constructing a certain trigonometric polynomial F whose Fourier coefficients have desired properties. One can notice that the difference between the two proofs is mainly due to the constants b_n in Theorem 2.2 (3), which lead us, naturally, to their definition. The Fejér kernels, used to define b_n, come from the construction of the trigonometric polynomial F and this is the departure point to the proof of Theorem 2.2.

I. Klemes in [6], as mentioned in chapter 1 page 20, constructed his F by using the recursive sequence (1.3). However, we, in the following proof of the existence of such F, are going to use another different construction, namely the recursive sequence (2.14). This construction is just a version of (1.2), the one used to prove the Littlewood conjecture [8]. The modification is mainly the convolution with Fejér kernels of certain orders. One now can understand the reason behind the definition of b_n.

The most important property of b_n is their absolute lower boundedness, and here comes the benefit of Proposition 2.1.

Proof of Proposition 2.1. Fix $n \geq 1$, choose j such that $n \in [4^{j-1}, 4^j)$, and define

$$\beta_L(n) = \prod_{\ell=j}^{L} \hat{K}_{(\ell)}(n), \qquad L \geq j.$$

since $0 < \hat{K}_{(\ell)}(n) < 1 \quad \forall\, \ell \geq j$, then $\{\beta_L(n)\}_{L=j}^{\infty}$ is a decreasing sequence, hence $B_n = \lim\limits_{L \to \infty} \beta_L(n)$ exists and is well-defined.

Since $\hat{K}_{(\ell)}(n) > 0 \quad \forall\, \ell \geq j$, then we can write

$$\beta_L(n) = \exp\left(\sum_{\ell=j}^{L} \ln\left(\hat{K}_{(\ell)}(n) \right) \right).$$

Now, we recall **Ineq. 2** on page 15, that is for real x, $0 \leq x \leq \frac{1}{2}$,

$$\ln(1-x) \geq -2x. \tag{2.13}$$

Since for $\ell \geq j$, $\hat{K}_{(\ell)}(n) = \left(1 - \frac{n}{2 \cdot 4^\ell - 2}\right)$, and

$$0 \leq \frac{n}{2 \cdot 4^\ell - 2} \leq \frac{4^j - 1}{2(4^j - 1)} = \frac{1}{2},$$

then for $L \geq j$, **Ineq. 2** gives

$$\begin{aligned}
\sum_{\ell=j}^{L} \ln\left(\hat{K}_{(\ell)}(n) \right) &= \sum_{\ell=j}^{L} \ln\left(1 - \frac{n}{2 \cdot 4^\ell - 2} \right) \\
&\geq -2 \sum_{\ell=j}^{L} \frac{n}{2 \cdot 4^\ell - 2} \\
&\geq -2 \sum_{\ell=j}^{L} \frac{4^j - 1}{2 \cdot 4^\ell - 2} \quad = -2 \sum_{\ell=0}^{L-j} \frac{4^j - 1}{2 \cdot 4^{j+\ell} - 2} \\
&\geq -2 \sum_{\ell=0}^{L-j} \frac{4^j - 1}{2 \cdot 4^j \cdot 4^\ell - 2 \cdot 4^\ell} \quad = -\sum_{\ell=0}^{L-j} \frac{4^j - 1}{4^\ell (4^j - 1)} \\
&\geq -\sum_{\ell=0}^{\infty} \frac{1}{4^\ell} \quad = -\frac{4}{3}.
\end{aligned}$$

Since e^x is an increasing function, then

$$\beta_L(n) = \exp\left(\sum_{\ell=j}^{L} \ln\left(\hat{K}_{(\ell)}(n)\right)\right) \geq e^{-4/3} > \frac{1}{4} \quad \forall\, L \geq j.$$

Hence, we get
$$B_n = \lim_{L\to\infty} \beta_L(n) > \frac{1}{4}.$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

We should turn our attention now to the proof of Theorem 2.2, and for that we need first to prove some prerequisite results.

Notation 2.1. For $g \in L_2(\mathbb{T})$, set

$$g^*(t) = \hat{g}(0) + 2\sum_{-\infty}^{-1} \hat{g}(n)e^{int}, \qquad t \in \mathbb{T}.$$

The next lemma is already mentioned in [8, p. 615].

Lemma 2.1. *If g is a real-valued function in $L_2(\mathbb{T})$, then g^* has the following properties:*

(1) $g^* \in L_2(\mathbb{T})$;

(2) $\mathbf{Re}\, g^* = g$;

(3) $\widehat{g^*}(n) = 0, \quad n > 0$;

(4) $\|g^*\|_2 \leq 2\|g\|_2$.

Proof. Since $g \in L_2(\mathbb{T})$; i.e. $\{\hat{g}(n)\}_{-\infty}^{0} \in \ell_2(\mathbb{Z})$, then $\left\{\widehat{g^*}(n)\right\} \in \ell_2(\mathbb{Z})$; consequently, by Riesz-Fischer theorem, g^* is well defined and belongs to $L_2(\mathbb{T})$. (3) is easily seen and (4) is clear by Parseval's theorem. Now for (2), observe that $\overline{\hat{g}(n)} = \hat{g}(-n)$, because g is a real-valued function, and hence

$$\begin{aligned}\mathbf{Re}\, g^*(t) &= \frac{1}{2}\left(g^*(t) + \overline{g^*(t)}\right)\\ &= \hat{g}(0) + \sum_{-\infty}^{-1}\left(\hat{g}(n)e^{int} + \overline{\hat{g}(n)}e^{-int}\right)\\ &= \sum_{-\infty}^{\infty} \hat{g}(n)e^{int} = g(t).\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Next, we prefer to recall some of the information and terminologies we have already used during the proof of the main result (Theorem 2.1). First we have $f \in L_1(\mathbb{T})$ is a trigonometric polynomial, moreover,

$$f(t) = \sum_{n=-4^J+1}^{4^J-1} \hat{f}(n) e^{int}, \qquad t \in \mathbb{T},$$

where J, which has been assumed to be ≥ 1, is the smallest non-negative integer such that $\operatorname{spec}(f) \subseteq (-4^J, 4^J)$. In addition, we have the trigonometric polynomials φ_j, $j = 1, \cdots, J$, defined by

$$\varphi_j(t) = \begin{cases} 0\,; & \text{if } a_n = 0 \quad \forall\, n \in I_j \\ 4^{-j/2}\left(\sum_{k\in I_j} |a_k|^2\right)^{-1/2} \sum_{n\in I_j} a_n e^{int}; & \text{otherwise} \end{cases}, \qquad t \in \mathbb{T},$$

where $a_n = \hat{f}(n)$ and $I_j = [4^{j-1}, 4^j)$.

It should be noted, however, that J is fixed throughout the rest of this section. Also, for future purposes, please keep the above information in mind.

Since $\varphi_j \in L_2(\mathbb{T})$, then we can define $h_j = |\varphi_j|^*$ for $j = 1, \cdots, J$.

Now, define the following inductive sequence of trigonometric polynomials

$$\begin{aligned} F_1^{(\epsilon)} &= K_{(1)} * \left(\varphi_1 e^{-\epsilon h_1}\right); \\ F_{j+1}^{(\epsilon)} &= K_{(j+1)} * \left(\left[F_j^{(\epsilon)} + \varphi_{j+1}\right] e^{-\epsilon h_{j+1}}\right), \qquad 1 \leq j \leq J, \end{aligned} \tag{2.14}$$

and put $F^{(\epsilon)} = F_J^{(\epsilon)}$, where $0 < \epsilon < 1$ is a parameter to be decided later.

We claim that this $F^{(\epsilon)}$ has enough power to attain the desired end and achieve our goal. First, we need to study the behaviour of the above inductive sequence. We start with the next lemma.

Lemma 2.2.

(1) $\left\|F^{(\epsilon)}\right\|_\infty \leq \frac{1}{\epsilon}$.

(2) For $1 \leq j \leq J$, $\operatorname{spec}\left(F_j^{(\epsilon)}\right) \subseteq [-2 \cdot 4^j + 3, 4^j)$.

Proof. (1) Since $F^{(\epsilon)} = F_J^{(\epsilon)}$, it is more than enough to show that for all $1 \le j \le J$,

$$\left| F_j^{(\epsilon)}(t) \right| \le \frac{1}{\epsilon} \quad \forall\, t \in \mathbb{T}.$$

First, since for $1 \le j \le J$, $h_j = |\varphi_j|^*$, then by Lemma 2.1 (2), $\mathbf{Re}\ h_j = |\varphi_j|$. Hence,

$$\left| e^{-\epsilon h_j(t)} \right| = e^{-\epsilon \mathbf{Re}\ h_j(t)} = e^{-\epsilon|\varphi_j(t)|} \le \frac{1}{1+\epsilon|\varphi_j(t)|} \quad \forall\, t \in \mathbb{T}, \tag{2.15}$$

where we have used the known inequality $e^{-x} \le \frac{1}{1+x}$, $x \ge 0$.

Therefore, for $1 \le j \le J$, we have

$$\left| \varphi_j(t) e^{-\epsilon h_j(t)} \right| \le \frac{|\varphi_j(t)|}{1+\epsilon|\varphi_j(t)|} \le \frac{1}{\epsilon} \quad \forall\, t \in \mathbb{T}. \tag{2.16}$$

Now consider the following mathematical induction. If $j = 1$ and $t \in \mathbb{T}$, we have

$$\begin{aligned}
\left| F_1^{(\epsilon)}(t) \right| &= \left| K_{(1)} * \left(\varphi_1 e^{-\epsilon h_1}\right)(t) \right| \\
&\le \int_{\mathbb{T}} K_{(1)}(t-\tau) \left| \varphi_1(\tau) e^{-\epsilon h_1(\tau)} \right| \frac{d\tau}{2\pi} \\
&\le \frac{1}{\epsilon} \int_{\mathbb{T}} K_5(t-\tau) \frac{d\tau}{2\pi} \\
&= \frac{1}{\epsilon} \int_{\mathbb{T}} K_5(\tau) \frac{d\tau}{2\pi} = \frac{1}{\epsilon},
\end{aligned}$$

where we have used the fact that Lebesgue measure is translation invariant on $\mathbb{T}$.

Suppose now that $\left| F_j^{(\epsilon)}(t) \right| \le \frac{1}{\epsilon}$ $\forall\, t \in \mathbb{T}$, for some fixed $1 \le j < J$. Then, by using the induction assumption and (2.16), we pass to the last step as follows. For $t \in \mathbb{T}$, we have

$$\begin{aligned}
\left| F_{j+1}^{(\epsilon)}(t) \right| &= \left| K_{(j+1)} * \left(\left[F_j^{(\epsilon)} + \varphi_{j+1} \right] e^{-\epsilon h_{j+1}} \right)(t) \right| \\
&\le \int_{\mathbb{T}} K_{(j+1)}(t-\tau) \left| \left[F_j^{(\epsilon)}(\tau) + \varphi_{j+1}(\tau) \right] e^{-\epsilon h_{j+1}(\tau)} \right| \frac{d\tau}{2\pi} \\
&\le \int_{\mathbb{T}} K_{(j+1)}(t-\tau) \left[\left| F_j^{(\epsilon)}(\tau) \right| + |\varphi_{j+1}(\tau)| \right] \left| e^{-\epsilon h_{j+1}(\tau)} \right| \frac{d\tau}{2\pi} \\
&\le \frac{1}{\epsilon} \int_{\mathbb{T}} K_{(j+1)}(t-\tau) \frac{1+\epsilon|\varphi_{j+1}(\tau)|}{1+\epsilon|\varphi_{j+1}(\tau)|} \frac{d\tau}{2\pi} = \frac{1}{\epsilon}.
\end{aligned}$$

(2) For $1 \le j \le J$, we know that φ_j is a trigonometric polynomial and hence it is in $L_2(\mathbb{T})$. Also, from (2.15), and from the proof of (1) in this lemma, we have $e^{-\epsilon h_j}$ and $F_j^{(\epsilon)} \in L_\infty(\mathbb{T}) \subseteq L_2(\mathbb{T})$. Therefore, we can use Lemma 0.1 in the following mathematical induction.

For $1 \le j \le J$, recall that $\operatorname{spec}(\varphi_j) \subseteq [4^{j-1}, 4^j)$; and it is clear by definition of h_j that $\operatorname{spec}(h_j) \subseteq (-\infty, 0]$; hence also $\operatorname{spec}\left(e^{-\epsilon h_j}\right) \subseteq (-\infty, 0]$. Therefore, we have

$$\begin{aligned} \operatorname{spec}\left(\varphi_1 e^{-\epsilon h_1}\right) &\subseteq \operatorname{spec}(\varphi_1) + \operatorname{spec}\left(e^{-\epsilon h_1}\right) \\ &\subseteq [1,4) + (-\infty, 0] = (-\infty, 4). \end{aligned}$$

Then
$$\begin{aligned} \operatorname{spec}\left(F_1^{(\epsilon)}\right) &= \operatorname{spec}\left(K_{(1)} * \left(\varphi_1 e^{-\epsilon h_1}\right)\right) \\ &= \operatorname{spec}(K_{(1)}) \cap \operatorname{spec}(\varphi_1 e^{-\epsilon h_1}) \\ &\subseteq [-5, 5] \cap (-\infty, 4) = [-5, 4). \end{aligned}$$

Now, suppose that $\operatorname{spec}\left(F_j^{(\epsilon)}\right) \subseteq [-2 \cdot 4^j + 3, 4^j)$ for fixed $1 \le j < J$. Then

$$\begin{aligned} \operatorname{spec}\left(F_j^{(\epsilon)} + \varphi_{j+1}\right) &\subseteq \operatorname{spec}\left(F_j^{(\epsilon)}\right) \cup \operatorname{spec}(\varphi_{j+1}) \\ &\subseteq [-2 \cdot 4^j + 3, 4^j) \cup [4^j, 4^{j+1}) \\ &= [-2 \cdot 4^j + 3, 4^{j+1}). \end{aligned}$$

Hence,
$$\begin{aligned} \operatorname{spec}\left(\left[F_j^{(\epsilon)} + \varphi_{j+1}\right] e^{-\epsilon h_{j+1}}\right) &\subseteq \operatorname{spec}\left(F_j^{(\epsilon)} + \varphi_{j+1}\right) + \operatorname{spec}\left(e^{-\epsilon h_{j+1}}\right) \\ &\subseteq [-2 \cdot 4^j + 3, 4^{j+1}) + (-\infty, 0] \\ &= (-\infty, 4^{j+1}). \end{aligned}$$

Therefore, the final step completes the proof as follows

$$\begin{aligned} \operatorname{spec}\left(F_{j+1}^{(\epsilon)}\right) &= \operatorname{spec}\left(K_{(j+1)} * \left(\left[F_j^{(\epsilon)} + \varphi_{j+1}\right] e^{-\epsilon h_{j+1}}\right)\right) \\ &= \operatorname{spec}\left(K_{(j+1)}\right) \cap \operatorname{spec}\left(\left[F_j^{(\epsilon)} + \varphi_{j+1}\right] e^{-\epsilon h_{j+1}}\right) \\ &\subseteq [-2 \cdot 4^{j+1} + 3, 2 \cdot 4^{j+1} - 3] \cap (-\infty, 4^{j+1}) \\ &= [-2 \cdot 4^{j+1} + 3, 4^{j+1}). \end{aligned}$$

The proof is done. $\mathbb{Q.E.D.}$

As a consequence of the nature of the above construction (2.14), we set the following notation, for convenience. First, note that if $g \in L_1(\mathbb{T})$, then $e^{-\epsilon h_j} g \in L_1(\mathbb{T})$ for all $1 \leq j \leq J$, because $\left|e^{-\epsilon h_j}\right| \leq 1$, by (2.15); hence we are allowed to do the following.

For $1 \leq m \leq M \leq J$, and $g \in L_1(\mathbb{T})$, set for "**C**onvolution with **F**ejér kernels",

$$\mathbf{CF}[M,m,g] = K_{(M)} * \left(e^{-\epsilon h_M} \left(K_{(M-1)} * \left(\cdots \left(e^{-\epsilon h_{m+1}} \left(K_{(m)} * \left(e^{-\epsilon h_m} g\right)\right)\right) \cdots\right)\right)\right). \quad (2.17)$$

For fixed M and $g \in L_1(\mathbb{T})$, the reader may regard $\mathbf{CF}[M,m,g]$ as a finite sequence of functions on m. One can also think of it as an infinite sequence of functions on m, via generalizing the notation for $m = 0$ and $m > M$.

First note that

(1) Since $K_{(m)} = K_{2\cdot 4^m - 3}$ $\quad m \geq 1$, we can write $K_{(0)} = 0$.

(2) $\mathbf{CF}[J,J,g] = K_{(J)} * \left(e^{-\epsilon h_J} g\right)$.

(3) For indices $m > J$, we can define $\varphi_m = 0$; and since $h_m = |\varphi_m|^*$, then $e^{-\epsilon h_m} = 1$.

Therefore, it makes sense to set the following generalized notation.

Notation 2.2. For $1 \leq M \leq J$, $m \geq 0$ and $g \in L_1(\mathbb{T})$, put

$$\mathbf{CF}[M,m,g] = \begin{cases} 0; & \text{if } m = 0 \\ \text{Formula (2.17)}; & \text{if } 1 \leq m \leq M \\ g; & \text{if } m > M \end{cases}.$$

Next, we register the following easily-proved properties of the above notation.

Proposition 2.2. *Let* $1 \leq M \leq J$, $m \geq 0$, $\alpha, \beta \in \mathbb{C}$, *and* $g, h \in L_1(\mathbb{T})$, *then*

(1) $\mathbf{CF}[M,m,\alpha g + \beta h] = \alpha \mathbf{CF}[M,m,g] + \beta \mathbf{CF}[M,m,h]$;

(2) $\mathbf{CF}[M,m,g] = \mathbf{CF}\left[M, m+1, K_{(m)} * \left(e^{-\epsilon h_m} g\right)\right], \quad 0 \leq m \leq M$;

(3) $\mathbf{CF}[M+1,m,g] = K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \mathbf{CF}[M,m,g]\right), \quad 0 \leq m \leq M+1 \leq J$.

Now, we are ready to show how the above notation plays the key role in many results that will simplify the proof of Theorem 2.2.

Lemma 2.3.

(1) For $1 \le m \le J$, $\operatorname{spec}(\mathbf{CF}[J, m, \varphi_m]) \subseteq [-2 \cdot 4^J + 3, 4^m)$.

(2) For $1 \le m \le J + 1$,

$$F^{(\epsilon)} = \mathbf{CF}\left[J, m-1, F^{(\epsilon)}_{m-2}\right] + \sum_{p=m-1}^{J} \mathbf{CF}[J, p, \varphi_p].$$

(Here, for convenience, put $F^{(\epsilon)}_{-1} = F^{(\epsilon)}_0 = \varphi_0 = 0$.*)*

(3) For $g \in L_1(\mathbb{T})$ *and* $1 \le m \le J$,

$$\begin{aligned}\mathbf{CF}[J, m, g] \;=\;& \mathbf{CF}\left[J, m+1, K_{(m)} * \left(\left(e^{-\epsilon h_m} - 1\right) g\right)\right] \\ &+ \sum_{p=m}^{J-1} \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left(\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * g\right)\right)\right] \\ &+ K_{(J)} * \cdots * K_{(m)} * g.\end{aligned}$$

Proof. (1) It is more than enough to prove that for all m and M such that $1 \le m \le M \le J$, we have

$$\operatorname{spec}\left(\mathbf{CF}[M, m, \varphi_m]\right) \subseteq [-2 \cdot 4^M + 3, 4^m).$$

We show that by mathematical induction on M. If $M = 1$, then

$$\begin{aligned}\operatorname{spec}\left(\mathbf{CF}[1,1,\varphi_1]\right) &= \operatorname{spec}\left(K_{(1)} * \left(e^{-\epsilon h_1}\varphi_1\right)\right) \\ &\subseteq \operatorname{spec}\left(K_{(1)}\right) \cap \left\{\operatorname{spec}\left(e^{-\epsilon h_1}\right) + \operatorname{spec}(\varphi_1)\right\} \\ &\subseteq [-5, 5] \cap \{(-\infty, 0] + [1, 4)\} \;=\; [-5, 4).\end{aligned}$$

Now, suppose that for fixed $M \le J - 1$,

$$\operatorname{spec}\left(\mathbf{CF}[M, m, \varphi_m]\right) \subseteq [-2 \cdot 4^M + 3, 4^m) \quad \forall\ 1 \le m \le M.$$

Then, from Proposition 2.2 (3), we have for all $1 \le m \le M + 1$,

$$\mathbf{CF}[M+1, m, \varphi_m] = K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \mathbf{CF}[M, m, \varphi_m]\right).$$

Hence, if $\Gamma = \text{spec}\,(\mathbf{CF}[M+1, m, \varphi_m])$, then

$$\begin{aligned}
\Gamma &\subseteq \text{spec}\left(K_{(M+1)}\right) \cap \left\{\text{spec}\left(e^{-\epsilon h_{M+1}}\right) + \text{spec}\,(\mathbf{CF}[M, m, \varphi_m])\right\} \\
&\subseteq [-2\cdot 4^{M+1}+3, 2\cdot 4^{M+1}-3] \cap \left\{(-\infty, 0] + [-2\cdot 4^{M}+3, 4^m)\right\} \\
&= [-2\cdot 4^{M+1}+3, 4^m).
\end{aligned}$$

(2) Following the same idea, it is more than enough to show that for all m and M such that $1 \le m \le M+1 \le J+1$, we have

$$F_M^{(\epsilon)} = \mathbf{CF}\left[M, m-1, F_{m-2}^{(\epsilon)}\right] + \sum_{p=m-1}^{M} \mathbf{CF}[M, p, \varphi_p].$$

Again we do it by mathematical induction on M. First, suppose $M = 1$; and recall that $F_{-1}^{(\epsilon)} = F_0^{(\epsilon)} = \varphi_0 = 0$. Now, if $m = 1$, we get

$$\mathbf{CF}\left[1, 0, F_{-1}^{(\epsilon)}\right] + \sum_{p=0}^{1} \mathbf{CF}[1, p, \varphi_p] = 0 + 0 + \mathbf{CF}[1, 1, \varphi_1] = K_{(1)} * \left(e^{-\epsilon h_1}\varphi_1\right) = F_1^{(\epsilon)}.$$

Similarly, if $m = 2$, we obtain

$$\mathbf{CF}\left[1, 1, F_0^{(\epsilon)}\right] + \sum_{p=1}^{1} \mathbf{CF}[1, p, \varphi_p] = 0 + \mathbf{CF}[1, 1, \varphi_1] = F_1^{(\epsilon)}.$$

So, suppose that our claim is true for some fixed $M \le J$. If $m = M+2$, then we are done, because

$$\begin{aligned}
F_{M+1}^{(\epsilon)} &= K_{(M+1)} * \left(\left[F_M^{(\epsilon)} + \varphi_{M+1}\right] e^{-\epsilon h_{M+1}}\right) \\
&= K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} F_M^{(\epsilon)}\right) + K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \varphi_{M+1}\right) \\
&= \mathbf{CF}\left[M+1, m-1, F_{m-2}^{(\epsilon)}\right] + \sum_{p=m-1}^{M+1} \mathbf{CF}[M+1, p, \varphi_p].
\end{aligned}$$

So, assume that $m \le M+1$. Then by the induction assumption, we have

$$\begin{aligned}
F_{M+1}^{(\epsilon)} &= K_{(M+1)} * \left(\left[F_M^{(\epsilon)} + \varphi_{M+1}\right] e^{-\epsilon h_{M+1}}\right) \\
&= K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \left\{\mathbf{CF}\left[M, m-1, F_{m-2}^{(\epsilon)}\right] + \sum_{p=m-1}^{M} \mathbf{CF}[M, p, \varphi_p]\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
& +K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \varphi_{M+1}\right) \\
= \;& K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \mathbf{CF}\left[M, m-1, F^{(\epsilon)}_{m-2}\right]\right) \\
& + \sum_{p=m-1}^{M} K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \mathbf{CF}[M, p, \varphi_p]\right) \\
& + \mathbf{CF}[M+1, M+1, \varphi_{M+1}] \\
= \;& \mathbf{CF}\left[M+1, m-1, F^{(\epsilon)}_{m-2}\right] + \sum_{p=m-1}^{M} \mathbf{CF}[M+1, p, \varphi_p] \\
& + \mathbf{CF}[M+1, M+1, \varphi_{M+1}] \\
= \;& \mathbf{CF}\left[M+1, m-1, F^{(\epsilon)}_{m-2}\right] + \sum_{p=m-1}^{M+1} \mathbf{CF}[M+1, p, \varphi_p],
\end{aligned}
$$

Where we have used Proposition 2.2 (3) in the third step.

(3) Let $g \in L_1(\mathbb{T})$ be any arbitrary function. First recall that for any $1 \leq j \leq J$, $e^{-\epsilon h_j} g \in L_1(\mathbb{T})$; hence, $K_{(j)} * \left(e^{-\epsilon h_j} g\right)$ and $K_{(J)} * g \in L_1(\mathbb{T})$.

Consider the following mathematical induction on $J - m$. If $m = J$, then by using the definition of $\mathbf{CF}[J, J, g]$, we have

$$
\begin{aligned}
\mathbf{CF}[J, J, g] &= K_{(J)} * \left(e^{-\epsilon h_J} g\right) \\
&= K_{(J)} * \left(\left(e^{-\epsilon h_J} - 1\right) g + g\right) \\
&= K_{(J)} * \left(\left(e^{-\epsilon h_J} - 1\right) g\right) + K_{(J)} * g \\
&= \mathbf{CF}\left[J, J+1, K_{(J)} * \left(\left(e^{-\epsilon h_J} - 1\right) g\right)\right] + K_{(J)} * g \\
&\quad + \sum_{p=J}^{J-1} \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left(\left(e^{-\epsilon h_{p+1}} - 1\right) \left(K_{(p)} * \cdots * K_{(J)} * g\right)\right)\right].
\end{aligned}
$$

Now, suppose it is true for fixed m, $1 < m \leq J$, and for any function $g \in L_1(\mathbb{T})$. Then, by using Proposition 2.2 (2) and (1), we get

$$
\begin{aligned}
\mathbf{CF}[J, m-1, g] &= \mathbf{CF}\left[J, m, K_{(m-1)} * \left(e^{-\epsilon h_{m-1}} g\right)\right] \\
&= \mathbf{CF}\left[J, m, K_{(m-1)} * \left(\left(e^{-\epsilon h_{m-1}} - 1\right) g + g\right)\right] \\
&= \mathbf{CF}\left[J, m, K_{(m-1)} * \left(\left(e^{-\epsilon h_{m-1}} - 1\right) g\right)\right] + \mathbf{CF}\left[J, m, K_{(m-1)} * g\right].
\end{aligned}
$$

Using the induction assumption for $\mathbf{CF}\left[J, m, K_{(m-1)} * g\right]$; and letting, for convenience, $CF = \mathbf{CF}[J, m-1, g]$, then we obtain that

$$\begin{aligned}
CF &= \mathbf{CF}\left[J, m, K_{(m-1)} * \left(\left(e^{-\epsilon h_{m-1}} - 1\right) g\right)\right] \\
&\quad + \mathbf{CF}\left[J, m+1, K_{(m)} * \left(\left(e^{-\epsilon h_m} - 1\right)\left(K_{(m-1)} * g\right)\right)\right] \\
&\quad + \sum_{p=m}^{J-1} \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left(\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * K_{(m-1)} * g\right)\right)\right] \\
&\quad + K_{(J)} * \cdots * K_{(m)} * K_{(m-1)} * g \\
&= \mathbf{CF}\left[J, m, K_{(m-1)} * \left(\left(e^{-\epsilon h_{m-1}} - 1\right) g\right)\right] \\
&\quad + \sum_{p=m-1}^{J-1} \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left(\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m-1)} * g\right)\right)\right] \\
&\quad + K_{(J)} * \cdots * K_{(m-1)} * g.
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Since I_j denotes the interval $[4^{j-1}, 4^j)$, for sake of simplicity, we let ${}^{-}I_j$ denote the interval $(-4^j, -4^{j-1}]$.

Corollary 2.1.

(1) For $1 \le m \le J$, $\quad \|K_{(J)} * \cdots * K_{(m-1)} * F^{(\epsilon)}_{m-2}\|_{\ell_2({}^{-}I_m)} = 0.$

(2) $F^{(\epsilon)} = \sum_{\ell=1}^{J} \mathbf{CF}[J, \ell, \varphi_\ell].$

(3) $F^{(0)} = F^{(\epsilon)}|_{\epsilon=0} = \sum_{\ell=1}^{J} K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell.$

Proof. (1) Fix m as above. From Lemma 2.2 (2), we have

$$\operatorname{spec}\left(F^{(\epsilon)}_{m-2}\right) \subseteq [-2 \cdot 4^{m-2} + 3, 4^{m-2}) \subset [-2 \cdot 4^p + 3, 2 \cdot 4^p - 3] \quad \forall\, p \ge m-1.$$

Hence,

$$\begin{aligned}
\operatorname{spec}\left(K_{(J)} * \cdots * K_{(m-1)} * F^{(\epsilon)}_{m-2}\right) &= \bigcap_{p=m-1}^{J} \operatorname{spec}(K_{(p)}) \cap \operatorname{spec}\left(F^{(\epsilon)}_{m-2}\right) \\
&= \operatorname{spec}\left(F^{(\epsilon)}_{m-2}\right) \subseteq [-2 \cdot 4^{m-2} + 3, 4^{m-2}).
\end{aligned}$$

Therefore, we are done because $\mathcal{I}_m \cap [-2 \cdot 4^{m-2} + 3, 4^{m-2}) = \emptyset$.

(2) Put $j = 2$ in Lemma 2.3 (2).

(3) It is obvious. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

The next lemma gives approximations to some terms involving the $\mathbf{CF}[\cdot,\cdot,\cdot]$.

Lemma 2.4. *Let g be a bounded function and $1 \le m \le J$. Then*

(1) $\|\,\mathbf{CF}[J, m, \varphi_m]\,\|_2 \le \|\varphi_m\|_2$.

(2) $\|\,\mathbf{CF}[J, m, g] - K_{(J)} * \cdots * K_{(m)} * g\,\|_2 \le 2\epsilon \|g\|_\infty \sum_{p=m}^{J} \|\varphi_p\|_2$.

***Proof*.** Bearing in mind the facts that $\|K_N\|_1 = 1 \quad \forall\, N \ge 1$, and for all $1 \le m \le J$, $|e^{-\epsilon h_m}| \le 1$, consider the following.

(1) It is more than enough to prove that for all m and M such that $1 \le m \le M \le J$, we have

$$\|\,\mathbf{CF}[M, m, \varphi_m]\,\|_2 \le \|\varphi_m\|_2.$$

Suppose $M = 1$, then, via Young's inequality, we have

$$\begin{aligned} \|\mathbf{CF}[1, 1, \varphi_1]\|_2 &= \left\| K_{(1)} * \left(e^{-\epsilon h_1}\varphi_1\right)\right\|_2 \\ &\le \|K_{(1)}\|_1 \left\|e^{-\epsilon h_1}\varphi_1\right\|_2 \le \|\varphi_1\|_2. \end{aligned}$$

Assume, if it is possible, that the claim is true for fixed $M < J$ and for all $1 \le m \le M$. Then, by Proposition 2.2 (3) and again Young's inequality, we obtain for all $1 \le m \le M+1$,

$$\begin{aligned} \|\,\mathbf{CF}[M+1, m, \varphi_m]\,\|_2 &= \left\| K_{(M+1)} * \left(e^{-\epsilon h_{M+1}} \mathbf{CF}[M, m, \varphi_m]\right)\right\|_2 \\ &\le \|K_{(M+1)}\|_1 \left\| e^{-\epsilon h_{M+1}} \mathbf{CF}[M, m, \varphi_m]\,\right\|_2 \\ &\le \|\,\mathbf{CF}[M, m, \varphi_m]\,\|_2 \le \|\varphi_m\|_2. \end{aligned}$$

Observe that (in the last line) if $m = M+1$, then $\mathbf{CF}[M, m, \varphi_m] = \varphi_m$.

(2) Fix m, $1 \le m \le J$. By Lemma 2.3 (3), we have

$$\mathbf{CF}[J,m,g] - K_{(J)} * \cdots * K_{(m)} * g = \mathbf{CF}\left[J, m+1, K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right] +$$
$$+ \sum_{p=m}^{J-1} \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left[\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * g\right)\right]\right]. \qquad (2.18)$$

We need here to record some clear facts about the h_m, $1 \leq m \leq J$. First, since $h_m = |\varphi_m|^*$, then $\mathbf{Re} h_m = |\varphi_m| \geq 0$. Hence, by using **Ineq. 3** on page 15 and Lemma 2.1 (4), we have

$$\left\|e^{-\epsilon h_m} - 1\right\|_2 \leq \epsilon \|h_m\|_2 \leq 2\epsilon \|\varphi_m\|_2. \qquad (2.19)$$

Now, for simplicity of writing, let $x = \mathbf{CF}\left[J, m+1, K_{(m)} * \left(\left(e^{-\epsilon h_m} - 1\right) g\right)\right]$. Then, by using Young's inequality, and the facts mentioned at beginning of the proof, we obtain

$$\begin{aligned}
\|x\|_2 &= \left\|K_{(J)} * \left(e^{-\epsilon h_J}\left(K_{(J-1)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right)\right\|_2 \\
&\leq \|K_{(J)}\|_1 \left\|e^{-\epsilon h_J}\left(K_{(J-1)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right\|_2 \\
&\leq \left\|e^{-\epsilon h_J}\left(K_{(J-1)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right\|_2 \\
&\leq \left\|K_{(J-1)} * \left(e^{-\epsilon h_{J-1}}\left(K_{(J-2)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right)\right\|_2 \\
&\leq \|K_{(J-1)}\|_1 \left\|e^{-\epsilon h_{J-1}}\left(K_{(J-2)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right\|_2 \\
&\leq \left\|e^{-\epsilon h_{J-1}}\left(K_{(J-2)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right)\right\|_2 \\
&\leq \left\|K_{(J-2)} * \left(\cdots\left(e^{-\epsilon h_{m+1}}\left(K_{(m)} * \left[\left(e^{-\epsilon h_m} - 1\right) g\right]\right)\right)\cdots\right)\right\|_2 \\
&\leq \cdots \leq \left\|\left(e^{-\epsilon h_m} - 1\right) g\right\|_2 \\
&\leq \|g\|_\infty \|e^{-\epsilon h_m} - 1\|_2 \leq 2\epsilon \|g\|_\infty \|\varphi_m\|_2,
\end{aligned}$$

where we have used (2.19) in the last line.

Similarly, following the same idea of the above estimation, and letting

$$y_p = \mathbf{CF}\left[J, p+2, K_{(p+1)} * \left[\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * g\right)\right]\right],$$

where $m \leq p \leq J-1$, we get that

$$\begin{aligned}
\|y_p\|_2 &= \left\|K_{(J)} * \left(\cdots\left(e^{-\epsilon h_{p+2}}\left(K_{(p+1)} * \left[\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * g\right)\right]\right)\right)\cdots\right)\right\|_2 \\
&\leq \cdots \leq \left\|\left(e^{-\epsilon h_{p+1}} - 1\right)\left(K_{(p)} * \cdots * K_{(m)} * g\right)\right\|_2
\end{aligned}$$

$$\begin{aligned}
&\leq \|K_{(p)} * \cdots * K_{(m)} * g\|_\infty \left\|e^{-\epsilon h_{p+1}} - 1\right\|_2 \\
&\leq 2\epsilon \|g\|_\infty \|\varphi_{p+1}\|_2,
\end{aligned}$$

where we have used the fact that $\|K_{(p)} * \cdots * K_{(m)} * g\|_\infty \leq \|g\|_\infty$. To confirm this fact notice that for any bounded function G and for all $N \geq 1$,

$$|K_N * G\,(\tau)| \leq \int_{\mathbb{T}} K_N(\tau - t)\|G\|_\infty \frac{dt}{2\pi} = \|G\|_\infty \quad \forall\, \tau \in \mathbb{T}.$$

This implies that $\|K_N * G\|_\infty \leq \|G\|_\infty$; consequently, the result follows by classical mathematical induction.

Finally, back to (2.18); apply the L_2-norm and substitute by the last two estimations.

$$\begin{aligned}
\|\mathbf{CF}[J,m,g] - K_{(J)} * \cdots * K_{(m)} * g\|_2 &= \left\| x + \sum_{p=m}^{J-1} y_p \right\|_2 \\
&\leq \|x\|_2 + \sum_{p=m}^{J-1} \|y_p\|_2 \\
&\leq 2\epsilon\|g\|_\infty\|\varphi_m\|_2 + \sum_{p=m}^{J-1} 2\epsilon\|g\|_\infty\|\varphi_{p+1}\|_2 \\
&= 2\epsilon\|g\|_\infty \sum_{p=m}^{J} \|\varphi_p\|_2.
\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Corollary 2.2. *Let g be a bounded function, $n \in \mathbb{N}$ and $1 \leq m \leq J$. Then*

$$\|\mathbf{CF}[J,m,g]\|_{\ell_2(\daleth_n)} \leq \|K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\daleth_n)} + 2\epsilon\|g\|_\infty \sum_{p=m}^{J} \|\varphi_p\|_2,$$

where $\daleth_n = (-4^n, -4^{n-1}]$.

Proof. Remembering Parseval's theorem, this is easy. If $x = \mathbf{CF}[J,m,g]$, then

$$\begin{aligned}
\|x\|_{\ell_2(\daleth_n)} &= \|\mathbf{CF}[J,m,g] - K_{(J)} * \cdots * K_{(m)} * g + K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\daleth_n)} \\
&\leq \|\mathbf{CF}[J,m,g] - K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\daleth_n)} + \|K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\daleth_n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \|\mathbf{CF}[J,m,g] - K_{(J)} * \cdots * K_{(m)} * g\|_2 + \|K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\mathcal{I}_n)} \\
&\leq \ 2\epsilon \|g\|_\infty \sum_{p=m}^{J} \|\varphi_p\|_2 + \|K_{(J)} * \cdots * K_{(m)} * g\|_{\ell_2(\mathcal{I}_n)}.
\end{aligned}
$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Finally, we can say that we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. We have already claimed before that our $F^{(\epsilon)}$, built from the inductive sequence (2.14), is the one that satisfies the three properties, after choosing the ϵ, of course. To show that consider the following.

(1) we have already proved that $\|F^{(\epsilon)}\|_\infty \leq \frac{1}{\epsilon}$ in Lemma 2.2 (1). So, $c_1 = \frac{1}{\epsilon} > 0$, obviously.

(2) Fix $j \geq 1$. By Lemma 2.2 (2),

$$
\operatorname{spec}\left(F^{(\epsilon)}\right) = \operatorname{spec}\left(F_J^{(\epsilon)}\right) \subseteq [-2 \cdot 4^J + 3, 4^J).
$$

Therefore, If $j \geq J+2$,

$$
\left(\sum_{n \in I_j} |\widehat{F^{(\epsilon)}}(-n)|^2\right)^{1/2} = 0.
$$

So, assume $1 \leq j \leq J+1$.

Now, if we let $m = j$ in Lemma 2.3 (2), then we have

$$
\begin{aligned}
\left(\sum_{n \in I_j} |\widehat{F^{(\epsilon)}}(-n)|^2\right)^{1/2} &= \ \left\|F^{(\epsilon)}\right\|_{\ell_2(\mathcal{I}_j)} \\
&= \ \left\|\mathbf{CF}\left[J, j-1, F_{j-2}^{(\epsilon)}\right] + \sum_{p=j-1}^{J} \mathbf{CF}[J,p,\varphi_p]\right\|_{\ell_2(\mathcal{I}_j)} \\
&\leq \ \left\|\mathbf{CF}\left[J, j-1, F_{j-2}^{(\epsilon)}\right]\right\|_{\ell_2(\mathcal{I}_j)} + \sum_{p=j-1}^{J} \|\mathbf{CF}[J,p,\varphi_p]\|_{\ell_2(\mathcal{I}_j)} . \quad (2.20)
\end{aligned}
$$

From the proof of Lemma 2.2 (1), $\left\|F_{j-2}^{(\epsilon)}\right\|_\infty < \frac{1}{\epsilon}$, $1 \leq j \leq J+1$. Hence, Corollary 2.2 gives

$$
\left\|\mathbf{CF}\left[J, j-1, F_{j-2}^{(\epsilon)}\right]\right\|_{\ell_2(\mathcal{I}_j)} \leq \left\|K_{(J)} * \cdots * K_{(j)} * F_{j-2}^{(\epsilon)}\right\|_{\ell_2(\mathcal{I}_j)} + 2\epsilon \left\|F_{j-2}^{(\epsilon)}\right\|_\infty \sum_{p=j-1}^{J} \|\varphi_p\|_2
$$

$$\begin{aligned}
&\le \quad 0 + 2\epsilon \cdot \frac{1}{\epsilon} \sum_{p=j-1}^{\infty} \|\varphi_p\|_2 \\
&= \quad 2 \sum_{p=j-1}^{\infty} 4^{-p/2} = 8 \cdot 4^{-j/2},
\end{aligned}$$

where we have used Corollary 2.1 (1) and (2.9) in the last two lines, respectively.

Also, from Lemma 2.4 (1), we have

$$\|\mathbf{CF}[J,p,\varphi_p]\|_{\ell_2(I_j)} \le \|\mathbf{CF}[J,p,\varphi_p]\|_2 \le \|\varphi_p\|_2 = 4^{-p/2}. \tag{2.21}$$

Substituting the last two estimates into (2.20), we get

$$\begin{aligned}
\left(\sum_{n \in I_j} |\widehat{F^{(\epsilon)}}(-n)|^2 \right)^{1/2} \quad &\le \quad 8 \cdot 4^{-j/2} + \sum_{p=j-1}^{J} 4^{-p/2} \\
&\le \quad 8 \cdot 4^{-j/2} + \sum_{p=j-1}^{\infty} 4^{-p/2} \\
&= \quad (8+4)4^{-j/2} = 12 \cdot 4^{-j/2}.
\end{aligned}$$

Hence, $c_2 = 12$.

(3) First recall Corollary 2.1 (2) and (3). Since $\mathrm{spec}(\varphi_\ell) \subseteq I_\ell,\ 1 \le \ell \le J$, then

$$\begin{aligned}
F^{(0)}(t) \quad &= \quad \sum_{\ell=1}^{J} \left(K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell \right)(t) \\
&= \quad \sum_{\ell=1}^{J} \sum_{n=-\infty}^{\infty} b_n \hat{\varphi}_\ell(n) e^{int} \\
&= \quad \sum_{\ell=1}^{J} \sum_{n \in I_\ell} b_n \hat{\varphi}_\ell(n) e^{int}.
\end{aligned}$$

So, for fixed $j,\ \ 1 \le j \le J$, and $n \in I_j$, we have

$$\left| \widehat{F^{(\epsilon)}}(n) - b_n \hat{\varphi}_j(n) \right| = \left| \widehat{F^{(\epsilon)}}(n) - \widehat{F^{(0)}}(n) \right|.$$

Hence, actually, we need to show that

$$\left\| F^{(\epsilon)} - F^{(0)} \right\|_{\ell_2(I_j)} \le \frac{1}{8} 4^{-j/2}.$$

First, use Corollary 2.1 (2) and (3) to write

$$\begin{aligned} F^{(\epsilon)} - F^{(0)} &= \sum_{\ell=1}^{J} \mathbf{CF}[J,\ell,\varphi_\ell] - \sum_{\ell=1}^{J} K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell \\ &= \sum_{\ell=1}^{J} \mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell. \end{aligned} \tag{2.22}$$

Now, observe that

$$\begin{aligned} \operatorname{spec}(K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell) &= \bigcap_{p=\ell}^{J} \operatorname{spec}(K_{(p)}) \cap \operatorname{spec}(\varphi_\ell) \\ &= \operatorname{spec}(\varphi_\ell) = [4^{\ell-1}, 4^{\ell}). \end{aligned}$$

So, by using Lemma 2.3 (1), we get

$$\operatorname{spec}(\mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell) \subseteq [-2\cdot 4^J + 3, 4^\ell).$$

This means that for $1 \le \ell \le j-1$ and for $n \in I_j = [4^{j-1}, 4^j)$, we have

$$\left(\mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell\right)\widehat{\ }(n) = 0.$$

Using this fact and (2.22), we obtain for $n \in I_j$

$$\left(F^{(\epsilon)} - F^{(0)}\right)\widehat{\ }(n) = \sum_{\ell=j}^{J} \left(\mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell\right)\widehat{\ }(n).$$

Now notice that, by the Cauchy-Schwarz inequality,

$$\|\varphi_\ell\|_\infty \le \sum_{n\in I_\ell} |\hat{\varphi}_\ell(n)| \le 4^{\ell/2}\|\varphi_\ell\|_2 = 1.$$

The above facts, with Lemma 2.4 (2) and (2.9), imply that

$$\begin{aligned} \left\|F^{(\epsilon)} - F^{(0)}\right\|_{\ell_2(I_j)} &= \left\|\sum_{\ell=j}^{J} \mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell\right\|_{\ell_2(I_j)} \\ &\le \sum_{\ell=j}^{J} \left\|\mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell\right\|_{\ell_2(I_j)} \\ &\le \sum_{\ell=j}^{J} \left\|\mathbf{CF}[J,\ell,\varphi_\ell] - K_{(J)} * \cdots * K_{(\ell)} * \varphi_\ell\right\|_2 \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\ell=j}^{J} 2\epsilon \|\varphi_\ell\|_\infty \sum_{p=\ell}^{J} \|\varphi_p\|_2 \\
&\le 2\epsilon \sum_{\ell=j}^{J} \sum_{p=\ell}^{\infty} 4^{-p/2} \\
&\le 4\epsilon \sum_{\ell=j}^{\infty} 4^{-\ell/2} = 8\epsilon \cdot 4^{-j/2}.
\end{aligned}
$$

So, by choosing $\epsilon = \frac{1}{64}$; and hence $c_1 = 64$, the proof is done; and now I can say that it is my pleasure to type the welcome symbol $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Reformulation of the Main Result in $B_{2\,1}^{-1/2}$

In this section, we show that the mixed-norm generalization of Hardy's inequality (Theorem 2.1, p. 23) can be reformulated in terms of the norm of the Besov space $B_{2\,1}^{-1/2}$. We start with the following proposition.

Proposition 2.3. *There is an absolute constant* $a > 0$ *such that for all functions* $f \in L_1(\mathbb{T})$,

$$
\sum_{j=1}^{\infty} \left(2^{-j} \sum_{2^{j-1} \le n < 2^j} |\hat{f}(n)|^2 \right)^{1/2} \le a\|f\|_1 + a \sum_{j=1}^{\infty} \left(2^{-j} \sum_{2^{j-1} \le n < 2^j} |\hat{f}(-n)|^2 \right)^{1/2}.
$$

Instead of proving (directly) the above proposition, we are going to prove the following lemma.

Lemma 2.5. *Proposition 2.3 is equivalent to Theorem 2.1, p. 23.*

Proof. For the sake of simplicity of writing, put $\ddot{I}_j = [2^{j-1}, 2^j)$ and $\ddot{\ddot{I}}_j = [4^{j-1}, 4^j)$.

The strategy of the proof is to show that for any arbitrary function $f \in L_1(\mathbb{T})$,

$$
\sum_{j=1}^{\infty} \left(2^{-j} \sum_{n \in \ddot{I}_j} |\hat{f}(n)|^2 \right)^{1/2} \simeq \sum_{j=1}^{\infty} \left(4^{-j} \sum_{n \in \ddot{\ddot{I}}_j} |\hat{f}(n)|^2 \right)^{1/2},
$$

and

$$\sum_{j=1}^{\infty}\left(2^{-j}\sum_{n\in \ddot{I}_j}|\hat{f}(-n)|^2\right)^{1/2} \simeq \sum_{j=1}^{\infty}\left(4^{-j}\sum_{n\in \ddot{\ddot{I}}_j}|\hat{f}(-n)|^2\right)^{1/2},$$

Since the proof should be independent of f, then it is enough to prove only the first equivalence.

Write $A=\sum_{j=1}^{\infty}A_j$, and $B=\sum_{j=1}^{\infty}B_j$, where

$$A_j=\left(2^{-j}\sum_{n\in \ddot{I}_j}|\hat{f}(n)|^2\right)^{1/2}, \quad \text{and} \quad B_j=\left(4^{-j}\sum_{n\in \ddot{\ddot{I}}_j}|\hat{f}(n)|^2\right)^{1/2}.$$

Since $A_j\geq 0 \quad \forall\, j\geq 1$, then $S_J:=\sum_{j=1}^{J}A_j$ is an increasing sequence; and hence its limit as $J\to\infty$ exists (which may be ∞). Therefore, any subsequence of $\{S_J\}$ also tends to the same limit as $J\to\infty$, in particular, S_{2J}. However, since

$$S_{2J}=\sum_{j=1}^{2J}A_j=\sum_{j=1}^{J}A_{2j-1}+A_{2j},$$

then

$$A=\lim_{J\to\infty}S_{2J}=\lim_{J\to\infty}\sum_{j=1}^{J}A_{2j-1}+A_{2j}=\sum_{j=1}^{\infty}A_{2j-1}+A_{2j}.$$

Now we claim that $A_{2j-1}+A_{2j}\simeq B_j \quad \forall\, j\geq 1$. To see that fix $j\geq 1$ and recall **Ineq. 1** on page 14, which is

$$\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}\leq\sqrt{2}\,\sqrt{a+b}, \quad a,b\geq 0.$$

Since all the terms here are non-negative real numbers, then, by using **Ineq. 1**, we get

$$\begin{aligned} A_{2j-1}+A_{2j} &= \left(2^{-2j+1}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2\right)^{1/2}+\left(2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2} \\ &\leq \sqrt{2}\left(2^{-2j+1}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{2}\left(2\cdot 2^{-2j}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+2\cdot 2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2}\\
&= 2\left(2^{-2j}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2}\\
&= 2\left(4^{-j}\left(\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)\right)^{1/2} = 2B_j,
\end{aligned}$$

because $\ddot{I}_{2j-1}\cup \ddot{I}_{2j} = [2^{2(j-1)},2^{j-1})\cup[2^{2j-1},2^{2j}) = [2^{2(j-1)},2^{2j}) = \ddot{\ddot{I}}_j$.

Also, by **Ineq. 1**,

$$\begin{aligned}
A_{2j-1}+A_{2j} &\geq \left(2^{-2j+1}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2}\\
&\geq \left(2^{-2j}\sum_{n\in \ddot{I}_{2j-1}}|\hat{f}(n)|^2+2^{-2j}\sum_{n\in \ddot{I}_{2j}}|\hat{f}(n)|^2\right)^{1/2} = B_j.
\end{aligned}$$

Hence, we have $B_j \leq A_{2j-1}+A_{2j}\leq 2B_j \quad \forall j\geq 1$. Consequently, $B\leq A\leq 2B$; i.e. $A\simeq B$. $\mathbb{Q.E.D.}$

For the next theorem, recall the Besov spaces section in chapter 0, in particular, the definition of $B_{2\,1}^{-1/2}$ and the kernels W_n, and their properties (Proposition 0.2). Also, recall that for $f\in L_1(\mathbb{T})$, the Riesz projections $\mathbb{P}_+$ and $\mathbb{P}_-$ are

$$(\mathbb{P}_+f)(t)=\sum_{n>0}\hat{f}(n)e^{int} \quad \text{and} \quad (\mathbb{P}_-f)(t)=\sum_{n<0}\hat{f}(n)e^{int}, \quad t\in\mathbb{T}.$$

Theorem 2.3. *Proposition 2.3 is equivalent to the following:*

"There is an absolute constant $b>0$ such that for all functions $f\in L_1(\mathbb{T})$,

$$\|\mathbb{P}_+f\|_{B_{2\,1}^{-1/2}}\leq b\|f\|_1+b\|\mathbb{P}_-f\|_{B_{2\,1}^{-1/2}}."$$

Proof. Since, by the definition of W_n, $\operatorname{spec}(W_n)=(2^{n-1},2^{n+1})$, $n>0$, and $\widehat{W_n}(k)=\widehat{W}_{-n}(-k) \quad \forall\, k,n>0$, then for $n<0$, we get $\operatorname{spec}(W_n)=(-2^{-n+1},-2^{-n-1})$.

Therefore, for $n < 0$, we have

$$\operatorname{spec}(W_n * \mathbb{P}_+ f) = \operatorname{spec}(W_n) \cap \operatorname{spec}(\mathbb{P}_+ f) \subseteq \operatorname{spec}(W_n) \cap (0, \infty) = \emptyset\,;$$

consequently, by Parseval's theorem, $\|W_n * \mathbb{P}_+ f\|_2 = 0 \quad \forall\, n < 0$. Hence,

$$\begin{aligned}\|\mathbb{P}_+ f\|_{B_{2\,1}^{-1/2}} &= \sum_{n\in\mathbb{Z}} 2^{-|n|/2}\|W_n * \mathbb{P}_+ f\|_2 = \sum_{n\geq 0} 2^{-n/2}\|W_n * \mathbb{P}_+ f\|_2 \\ &= \sum_{n=0}^{\infty} \left(2^{-n}\|W_n * \mathbb{P}_+ f\|_2^2\right)^{1/2}.\end{aligned}$$

Now, since $0 \leq \widehat{W}_n(k) \leq 1 \quad \forall\, n, k \in \mathbb{Z}$, then, by Parseval's theorem, we obtain for $n > 0$,

$$\begin{aligned}\|W_n * \mathbb{P}_+ f\|_2^2 &= \sum_{k\in\mathbb{Z}} |\widehat{W}_n(k)|^2\, |\widehat{\mathbb{P}_+ f}(k)|^2 \\ &\leq \sum_{k=2^{n-1}}^{2^{n+1}} |\widehat{W}_n(k)|\, |\hat{f}(k)|^2.\end{aligned}$$

Also, since $W_0(t) = e^{-it} + 1 + e^{it}$, then

$$\begin{aligned}\|W_0 * \mathbb{P}_+ f\|_2^2 &= |\widehat{W}_0(-1)|^2\, |\widehat{\mathbb{P}_+ f}(-1)|^2 + |\widehat{W}_0(0)|^2\, |\widehat{\mathbb{P}_+ f}(0)|^2 + |\widehat{W}_0(1)|^2\, |\widehat{\mathbb{P}_+ f}(1)|^2 \\ &= |\hat{f}(1)|^2.\end{aligned}$$

For convenience, let $x = \|\mathbb{P}_+ f\|_{B_{2\,1}^{-1/2}}$. Then by using **Ineq. 1**, (0.4) and the last two estimations, we get

$$\begin{aligned}x &= \sum_{n=0}^{\infty} \left(2^{-n}\|W_n * \mathbb{P}_+ f\|_2^2\right)^{1/2} \\ &= \left(2^{0}\|W_0 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \left(2^{-1}\|W_1 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \left(2^{-2}\|W_2 * \mathbb{P}_+ f\|_2^2\right)^{1/2} \\ &\quad + \left(2^{-3}\|W_3 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \cdots \\ &\leq \left(|\hat{f}(1)|^2\right)^{1/2} + \left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} \\ &\quad + \left(2^{-2}\left(\frac{1}{2}|\hat{f}(3)|^2 + |\hat{f}(4)|^2 + \frac{3}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{1}{4}|\hat{f}(7)|^2\right)\right)^{1/2} \\ &\quad + \left(2^{-3}\left(\frac{1}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{3}{4}|\hat{f}(7)|^2 + \cdots + \frac{1}{8}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots\end{aligned}$$

$$
\begin{aligned}
&\leq \left(|\hat{f}(1)|^2\right)^{1/2} + \left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-2}\left(\frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} + \left(2^{-2}\left(|\hat{f}(4)|^2 + \frac{3}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{1}{4}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-3}\left(\frac{1}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{3}{4}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-3}\left(|\hat{f}(8)|^2 + \cdots + \frac{1}{8}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&\leq \left(|\hat{f}(1)|^2\right)^{1/2} + \left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-1}\left(\frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} + \left(2^{-2}\left(|\hat{f}(4)|^2 + \frac{3}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{1}{4}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-2}\left(\frac{1}{4}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{1}{4}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-3}\left(|\hat{f}(8)|^2 + \cdots + \frac{1}{4}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&\leq \sqrt{2}\left(|\hat{f}(1)|^2\right)^{1/2} + \sqrt{2}\left(2^{-1}\left(|\hat{f}(2)|^2 + |\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad \left(2^{-2}\left(|\hat{f}(4)|^2 + |\hat{f}(5)|^2 + |\hat{f}(6)|^2 + |\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad \left(2^{-3}\left(|\hat{f}(8)|^2 + \cdots + |\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&= 2\sqrt{2}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(k)|^2\right)^{1/2}.
\end{aligned}
$$

So, we have

$$\|\mathbb{P}_+ f\|_{B_{21}^{-1/2}} \leq 2^{3/2}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(k)|^2\right)^{1/2}.$$

Next, we claim that

$$\|\mathbb{P}_+ f\|_{B_{21}^{-1/2}} \geq 2^{-3/2}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(k)|^2\right)^{1/2}.$$

We do this, again by using the above estimation and **Ineq. 1**, as follows; recall that $x = \|\mathbb{P}_+ f\|_{B_{2\,1}^{-1/2}}$.

$$\begin{aligned}
x &= \sum_{n=0}^{\infty} \left(2^{-n}\|W_n * \mathbb{P}_+ f\|_2^2\right)^{1/2} \\
&= \left(2^{0}\|W_0 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \left(2^{-1}\|W_1 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \left(2^{-2}\|W_2 * \mathbb{P}_+ f\|_2^2\right)^{1/2} \\
&\quad + \left(2^{-3}\|W_3 * \mathbb{P}_+ f\|_2^2\right)^{1/2} + \cdots \\
&= \left(|\hat{f}(1)|^2\right)^{1/2} + \left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{2^2}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-2}\left(\frac{1}{2^2}|\hat{f}(3)|^2 + |\hat{f}(4)|^2 + \frac{3^2}{4^2}|\hat{f}(5)|^2 + \frac{1}{2^2}|\hat{f}(6)|^2 + \frac{1}{4^2}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \left(2^{-3}\left(\frac{1}{4^2}|\hat{f}(5)|^2 + \frac{1}{2^2}|\hat{f}(6)|^2 + \frac{3^2}{4^2}|\hat{f}(7)|^2 + \cdots + \frac{1}{8^2}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&\geq \left(|\hat{f}(1)|^2\right)^{1/2} + \left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{4}|\hat{f}(3)|^2\right)\right)^{1/2} + \frac{1}{\sqrt{2}}\left(2^{-2}\left(\frac{1}{4}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-2}\left(|\hat{f}(4)|^2 + \frac{9}{16}|\hat{f}(5)|^2 + \frac{1}{4}|\hat{f}(6)|^2 + \frac{1}{16}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-3}\left(\frac{1}{16}|\hat{f}(5)|^2 + \frac{1}{4}|\hat{f}(6)|^2 + \frac{9}{16}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad \frac{1}{\sqrt{2}}\left(2^{-3}\left(|\hat{f}(8)|^2 \cdots + \frac{1}{64}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&\geq \frac{1}{\sqrt{2}}\left(|\hat{f}(1)|^2\right)^{1/2} + \frac{1}{\sqrt{2}}\left(2^{-1}\left(|\hat{f}(2)|^2 + \frac{1}{4}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-2}\left(\frac{1}{4}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-2}\left(|\hat{f}(4)|^2 + \frac{9}{16}|\hat{f}(5)|^2 + \frac{1}{4}|\hat{f}(6)|^2 + \frac{1}{16}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-3}\left(\frac{1}{16}|\hat{f}(5)|^2 + \frac{1}{4}|\hat{f}(6)|^2 + \frac{9}{16}|\hat{f}(7)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-3}\left(|\hat{f}(8)|^2 \cdots + \frac{1}{64}|\hat{f}(15)|^2\right)\right)^{1/2} + \cdots \\
&\geq \frac{1}{\sqrt{2}}\left(\frac{1}{2}|\hat{f}(1)|^2\right)^{1/2} + \frac{1}{\sqrt{2}}\left(2^{-1}\left(\frac{1}{2}|\hat{f}(2)|^2 + \frac{1}{2}|\hat{f}(3)|^2\right)\right)^{1/2} \\
&\quad + \frac{1}{\sqrt{2}}\left(2^{-2}\left(\frac{1}{2}|\hat{f}(4)|^2 + \frac{1}{2}|\hat{f}(5)|^2 + \frac{1}{2}|\hat{f}(6)|^2 + \frac{1}{2}|\hat{f}(7)|^2\right)\right)^{1/2}
\end{aligned}$$

$$+\frac{1}{\sqrt{2}}\left(2^{-3}\left(\frac{1}{2}|\hat{f}(8)|^2\cdots+\frac{1}{2}|\hat{f}(15)|^2\right)\right)^{1/2}+\cdots$$

$$\geq \frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(k)|^2\right)^{1/2}.$$

Similarly, since $W_n(k)=W_{-n}(-k)\quad\forall\, k,n>0$, and the above proof is independent of f, we get

$$\begin{aligned}2^{-3/2}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(-k)|^2\right)^{1/2} &\leq \|\mathbb{P}_-f\|_{B_{2\,1}^{-1/2}}\\ &\leq 2^{3/2}\sum_{n=1}^{\infty}\left(2^{-n}\sum_{2^{n-1}\leq k<2^n}|\hat{f}(k)|^2\right)^{1/2}.\end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Corollary 2.3. *The mixed-norm generalization of Hardy's inequality can be reformulated in terms of the norm of* $B_{2\,1}^{-1/2}$ *as follows:*

"There is an absolute constant $b>0$ *such that for all functions* $f\in L_1(\mathbb{T})$*,*

$$\|\mathbb{P}_+f\|_{B_{2\,1}^{-1/2}}\leq b\|f\|_1+b\|\mathbb{P}_-f\|_{B_{2\,1}^{-1/2}}.\text{''}\tag{2.23}$$

Notice the similarity between (2.23) and (1.6) when $p=2$. Corollary 2.3 solves the reconstruction problem for the Besov space $B_{2\,1}^{-1/2}$; that is,

$$\mathbb{P}_-f\in B_{2\,1}^{-1/2}\Longrightarrow f\in B_{2\,1}^{-1/2}.$$

Now it is natural to raise the following question: for which numbers $p\geq 1$,

$$\mathbb{P}_-f\in B_{p\,1}^{-1/p}\Longrightarrow f\in B_{p\,1}^{-1/p}?$$

Conclusion

From this thesis, it can be concluded that the mixed-norm generalization of Hardy's inequality (Theorem 2.1) can be proved by using at least two different constructions namely (1.3) and (1.2). In addition, reformulation of the mixed-norm generalization can be presented in terms of the norm of the Besov space $B_{2\,1}^{-1/2}$. This reformulation solves the reconstruction problem of the Besov spaces $B_{2\,1}^{-1/2}$. In addition, all of these results can be restated in terms of finite complex Borel measures.

We conclude this study with the following questions, which may raise a suggestion for pursuing this work in the same direction:

- Is there any relationship between the mixed-norm generalization of Hardy's inequality and the open problem in (0.1)?
- Is it possible to rewrite the mixed-norm generalization of Hardy's inequality as follows:

$$\left| \sum_{j=1}^{\infty} \left(4^{-j} \sum_{4^{j-1} \leq n < 4^j} |\hat{f}(n)|^2 \right)^{1/2} - \sum_{j=1}^{\infty} \left(4^{-j} \sum_{4^{j-1} \leq n < 4^j} |\hat{f}(-n)|^2 \right)^{1/2} \right| \leq c \|f\|_1 \,?$$

- For which numbers $p \geq 1$,

$$\mathbb{P}_- f \in B_{p\,1}^{-1/p} \Longrightarrow f \in B_{p\,1}^{-1/p}?$$

- Can the mixed-norm generalization be deduced from Theorem 1.2?
- Can the mixed-norm generalization be deduced from Theorem 1.3?

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