Modified Quantized input Variable Step Size LMS, QX-VSS LMS Algorithm Applied to Signal Prediction

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***Abstract* — Several modified LMS algorithms are studied in order to improve the rate of convergence, increase the tracking performance and reduce the computational cost of the regular LMS algorithm. These methods can be divided in two categories: Clipped data algorithms and variable step size algorithms. In this paper a new quantized input variable step size LMS algorithm is introduced. The proposed algorithm is a modification of an existing method, namely, VSS LMS, and uses a new quantization function for clipping the input signal. We showed mathematically the convergence of the QX-VSS LMS filter weights to the optimum Wiener filter weights. Also, we proved that the proposed algorithm has better tracking than the conventional LMS algorithm. We discuss the conditions which one have to consider so that he can get better performance of QX-VSS LMS algorithm. The results of simulations confirm the presented mathematical analysis.**

***Index Terms* — Least mean square(LMS), Variable step size LMS, Weiner weights, Tracking.**

I. Introduction

The least mean square (LMS) algorithm [6] has been widely used in adaptive filtering because of its stability and simplicity of implementation. It has attained its popularity due to a broad range of useful applications in such diverse areas as communications, radar, sonar, seismology, navigation and control systems, and biomedical electronics.

But its convergence rate is slow and the performance varies depending on the statistical characteristics of an input signal [2]. So, a great number of algorithms have been proposed to speed up the convergence and reduce the computational time. These algorithms can be divided in two main categories: Clipped data algorithms and variable step size algorithms.

In Clipped data algorithms the clipping of input signal or error signal is a common approach to improving efficiency of LMS filters. Reduction of the complexity of the LMS algorithm has received attention in the area of adaptive filters [3, 5, 8, 9]. The sign algorithm and clipped data algorithms are in this category [1, 4, 3, 7, 8, 9].

The choice of step size reflects a tradeoff between missadjustment and the speed of adaptation [10]. Variable step size algorithms are another category of modified LMS algorithms that are simple to implement and are capable of giving both fast tracking as well as small missadjustment [10,11].In variable step size LMS algorithms, the step size, which controls convergence speed and stability after convergence, is updated as the iteration time changes.

In this paper a new algorithm called the quantized input variable step size LMS or QX-VSS LMS algorithm is introduced which is based on clipping input signal with a new proposed tgn(.) function and Kwong VSS LMS algorithm [10], whose tracking is much better than the VSS LMS and LMS and has less computation as well.

The variants of LMS are discussed in Section 2. The proposed new algorithm, which is a modification of the aforementioned algorithm, appears in Section 3. Section 4 deals with computer simulation issues. The final section presents conclusion and summarizes the main findings.

II. LMS Algorithm Variations

In this section we review the Standard LMS and the VSS LMS Algorithm [10] which is the foundation of our proposed algorithm.

A. Standard LMS Algorithm

The least mean square (LMS) algorithm [6] has been widely used in adaptive filtering because of its stability and simplicity of implementation. The LMS algorithm has been studied in [12] as:

$$W_{k+1} = W_k + \mu e_k X_k \qquad (1)$$

Where

$$e_k = d_k - X_k^T W_k \qquad (2)$$

$W_k = [w_k(1), w_k(2),..., w_k(N)]^T$ is the weight vector of the predictor, X_k is the vector of the input data sequence, which is assumed to be a stationary random process, N is the number of filter tapes, e_k is the estimation error, d_k is the desired response and μ is the step size.

B. VSS LMS Algorithm

Kwong and Johnston [10] proposed the VSS LMS algorithm in which the step size varies based on the error signal at every iteration. The filter weight updating formula at each iteration time is given by:

$$W_{k+1} = W_k + \mu_k e_k X_k \quad (3)$$

In which the step size, μ_k, is time varying with its value determined by the following equation:

$$\mu'_{k+1} = \alpha\mu_k + \gamma e_k^2 \quad (4)$$

With $0 < \alpha < 1$, $\gamma > 0$ And

$$\mu_{k+1} = \begin{cases} \mu_{\max} & if\ \mu'_{k+1} > \mu_{\max} \\ \mu_{\min} & if\ \mu'_{k+1} < \mu_{\min} \\ \mu'_{k+1} & otherwise \end{cases} \quad (5)$$

Where $0 < \mu_{\min} < \mu_{\max}$. As can be seen, the step size is controlled by the size of prediction error and the parameters α and γ. The initial step size μ_0 is usually taken to be $\mu_{\max}$, although the algorithm is not sensitive to the choice. A sufficient condition for $\mu_{\max}$ to guarantee bounded MSE is [13]:

$$\mu_{\max} \le \frac{2}{3tr(R)} \quad (6)$$

$\mu_{\min}$ is chosen to provide a minimum level of tracking ability.

III. Proposed QX-VSS LMS Algorithm

Here we propose a new modification to the VSS LMS algorithm [10] to further simplify the implementation of the LMS and decrease the computational complexity of it.

We performed the quantization on the input vector of VSS algorithm. In Fig.1 our proposed function to quantizing the input vector is shown.

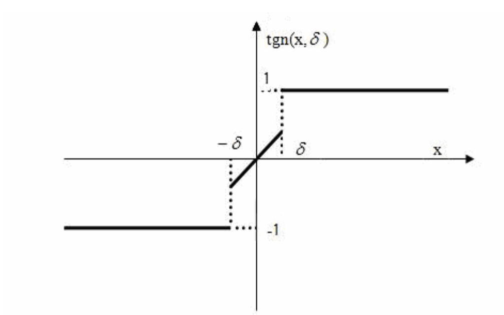

Fig. 1. Quantization scheme for the proposed algorithm

This function is defined as follows:

$$tsg(x,\delta) = \begin{cases} +1 & x > \delta \\ x & -\delta < x < \delta \\ -1 & x < -\delta \end{cases} \quad (7)$$

Consequently, the weight updating formula of QX-VSS LMS algorithm has the following form:

$$W_{k+1} = W_k + \mu_k e_k \widetilde{X}_k \quad (8)$$

Where $\widetilde{X}_k$ is the modified quantized input signal vector whose ith component is $\widetilde{x}_k(i) = tgn(x_k(i),\delta)$, and μ_k is the step size which varies according to (4) and (5). In the next sections we will discuss the convergence and tracking properties of proposed QX-VSS LMS algorithm. We will prove mathematically that the quantization of the input signal in noisy environments increases tracking power of the filter and decreases the computational cost of the algorithm.

The following subsections discuses mathematically the convergence of QX-VSS LMS weights to Weiner weights and its tracking performance.

A. Convergence of QX-VSS LMS Algorithm

It is usual in adaptive filter literatures to prove the convergence of the filter weights to Weiner optimum weights. Theorem 1 proves the convergence of QX-VSS LMS weights to optimum Weiner weights.

Theorem 1. *If the QX-VSS LMS weights, W_n, is described by the equation 8 and W_o is the Weiner optimum weight ,then W_n converges to W_o.*

Proof. For convergence prove it sufficient to show that: $\lim_{k\to\infty} E\{W_{k+1}\} = W_o$

Substituting e_k from Equation 2 into the Equation 2 yields: $W_{k+1} = W_k + \mu_k(d_k\widetilde{X}_k - \widetilde{X}_k X_k^T W_k)$

Regarding to expectation of this equation we have:

$$E\{W_{k+1}\} = E\{W_k\} + E\{\mu_k\}E\{d_k\widetilde{X}_k - \widetilde{X}X_k^T W_k\}$$

Assuming lack of correlation between the W_k and $\widetilde{X}_k X_k^T$ as in [1,7] , we have:

$$E\{W_{k+1}\} = E\{W_k\} + E\{\mu_k\}(E\{d_k\widetilde{X}_k\} - E\{\widetilde{X}X_k^T\}E\{W_k\})$$

With regard to Lemma 1 (see Appendix), $P = E\{d_k X_k\}$ and $R = E\{X_k X_k^T\}$ we have:

$$E\{W_{k+1}\} = E\{W_k\} + E\{\mu_k\}(\frac{\alpha'}{\sigma_x}P - \frac{\alpha'}{\sigma_x}.R.E\{W_k\})$$
$$= (I - E\{\mu_k\}\frac{\alpha'}{\sigma_x}R)E\{W_k\} + E\{\mu_k\}\frac{\alpha'}{\sigma_x}P \quad (9)$$

Replacing the Wiener optimum weight [6], $W^* = R^{-1}P$ in the Equation 9 we have:

$$E\{W_{k+1}\} = (I - E\{\mu_k\}\frac{\alpha'}{\sigma_x}R)E\{W_k\} + E\{\mu_k\}\frac{\alpha'}{\sigma_x}R.W^*$$

So, based on above relation the error weight vector [10] $V_k = W_k - W^*$ is calculated as:

$$E\{V_{k+1}\} = (I - E\{\mu_k\}.\frac{\alpha'}{\sigma_x}.R)E\{V_k\}$$

Since R, The covariance matrix of X_k, is symmetric, there exist matrices Q and Λ, with Λ diagonal, such that $R = Q\Lambda Q^T$ and $Q^T Q = I$ [10]. Let $V_k = QV'_k$ be the rotation of V'_k by Q , therefore we have the following relation: $E\{QV'_{k+1}\} = (I - E\{\mu_k\}\frac{\alpha'}{\sigma_x}.R)E\{QV'_k\}$

Where Q and V'_k are uncorrelated because W and R are uncorrelated. Thus:

$$E\{V'_{k+1}\} = Q^{-1}(I - E\{\mu_k\}\frac{\alpha'}{\sigma_x}.R).Q..E\{V'_k\}$$

$$E\{V'_{k+1}\} = (Q^{-1}IQ - E\{\mu_k\}\frac{\alpha'}{\sigma_x}Q^{-1}RQ)E\{V'_k\}$$

So, we have:

$$E\{V'_{k+1}\} = (I - E\{\mu_k\}\frac{\alpha'}{\sigma_x}\Lambda)E\{V'_k\} \quad (10)$$

The equation 10 can be written as:

$$E\{V'_{k+1}\} = \prod_{j=1}^{k}(I - E\{\mu_j\}\frac{\alpha'}{\sigma_x}\Lambda)^j E\{V'_1\}$$

Therefore, for showing the convergence of QX-VSS algorithm, it is sufficient to prove that the error weight vector (V_k), in the limit converges to zero. In other words we must demonstrate $Lim_{k\to\infty}E\{V_k\} = 0$. Regarding to relation of V'_k and V_k , $V_k = QV'_k$, proving $Lim_{k\to\infty}E\{V'_k\} = 0$ implies $Lim_{k\to\infty}E\{V_k\} = 0$.

We have:

$$Lim_{k\to\infty}E\{V'_k\} = \underset{k\to\infty}{Lim}\prod_{j=1}^{k}(I - E\{\mu_j\}\frac{\alpha'}{\sigma_x}\Lambda)^j E\{V'_1\}$$

We know that Λ is diagonal matrix. Suppose that the diagonal element of matrix Λ in i^th column denoted by λ_i , for $i = 1,...,N$. We have:

$$0 < E\{\mu_j\}\frac{\alpha'}{\sigma_x}\lambda_i < 1 \Rightarrow 0 < E\{\mu_j\} < \frac{\sigma_x}{\alpha'}\frac{1}{\lambda_i} \qquad , for\ j = 1,...,k$$

If $E\{\mu_i\}$ satisfies this relation for the largest $\lambda_{\max}$, then above relation is also satisfied for all others. Thus, the convergence condition for proposed QX-VSS LMS algorithm is as follow:

$$0 < E\{\mu_j\} < \frac{\sigma_x}{\alpha'}.\frac{1}{\lambda_{\max}} \qquad , for\ j = 1,...,k \quad (11)$$

Hence, the proof is complete.☐

B. Tracking Power of QX-VSS LMS Algorithm

Tracking in filter theory, means the tracking of filter weights. According to [1, 13], Tracking is a steady-state phenomenon that is different from the convergence, which is a transient phenomenon. In general, convergence and tracking are two different properties. That is, if an algorithm has good convergence, its tracking ability is not necessarily fast and vice versa. In the tracking phase, a reasonable assumption is that the optimum weights vary according to a first-order Markov process [14], and the filter must track these weights. The following relation shows the variation of the filter's optimum weights:

$$W^*_{n+1} = aW^*_n + \omega_n$$
$$d_n = W_n^{*T} X_n + \upsilon_n \quad (12)$$

Where a is a constant and ω_n is the process noise vector in the nth step, which has zero mean, and υ_n is the measurement noise, which is assumed to be white Gaussian with zero mean and variance σ_v^2 .

Missadjustment criterion as shown in Equation 13 is a usable measure for tacking performance [14]:

$$Misadjusment = \frac{E\{|\omega_n^T X_n|^2\}}{E\{|\upsilon_n|^2\}} \quad (13)$$

In the following theorem we will show the relation between our new proposed QX-VSS LMS and the VSS LMS according to this criterion.

Theorem 2. *Let M_{QX-VSS} and M_{VSS} are QX-VSS LMS and conventional VSS LMS miss adjustment criterion, respectively. Then: $M_{QX-VSS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{VSS}$*

Where: $\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_x^2) + \sigma_x erf(\delta/\sqrt{2}\,\sigma_x)$.

Proof.

$$E\{|\omega_n^T \widetilde{X}_n|^2\} = E\{\omega_n^T \widetilde{X}_n \widetilde{X}_n^T \omega_n\}$$

With suppose the independency between ω_n and $\widetilde{X}_n$ and by using Lemma 1 we have:

$$E\{|\omega_n^T \widetilde{X}_n|^2\} = E\{\omega_n^T \widetilde{X}_n \widetilde{X}_n^T \omega_n\} = E\{\omega_n^T \widetilde{X}_n\}E\{\widetilde{X}_n^T \omega_n\}$$
$$= \left(\frac{\alpha'}{\sigma_x}E\{\omega_n^T \widetilde{X}_n\}\right)\left(\frac{\alpha'}{\sigma_x}E\{\widetilde{X}_n^T \omega_n\}\right) = \left(\frac{\alpha'}{\sigma_x}\right)^2 E\{|\omega_n^T X_n|^2\}$$

By dividing the equation to $E\{|\upsilon_n|^2\}$:

$$\frac{E\{|\omega_n^T \widetilde{X}_n|^2\}}{E\{|\upsilon_n|^2\}} = \left(\frac{\alpha'}{\sigma_x}\right)^2 \frac{E\{|\omega_n^T X_n|^2\}}{E\{|\upsilon_n|^2\}} \Rightarrow_{QX-VSS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{VSS}$$

and hence the proof is complete.☐

In an instance filter, the reducing of the missadjustment measure means the increasing of the tracking performance [14]. Hence, from theorem 2 it is obvious that if $\alpha' < \sigma_x$, tracking performance of QX-VSS LMS is better than VSS LMS. But it does not discuss conditions which one have to consider so that he can get better performance of QX-VSS LMS algorithm. Theorem 3 describes the suitable parameter values.

Theorem 3. *If $\delta = \sqrt{2}\sigma_x$ then tracking performance of QX-VSS LMS algorithm is better than conventional VSS LMS algorithm for $\sigma_x > 0.513$. Where σ_x is the variance of input signal, and δ is the parameter of $tgn(x,\delta)$.*

Proof. It is obvious that, if the missadjustment of QX-VSS LMS is lower than the missadjustment of conventional VSS LMS, then the tracking performance of QX-VSS LMS is better than VSS LMS. Regarding to theorem 2 results, we have:

$$M_{QX-VSS} = \left(\frac{\alpha'}{\sigma_x}\right)^2 M_{VSS}$$

Where $\alpha' = \sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2) + \sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$

Now, it is sufficient to show that if $\delta = \sqrt{2}\sigma_x$ then $\alpha' < \sigma_x$, hence:

$$\alpha' = \sqrt{2/\pi}(1-\sqrt{2}\sigma_x)\exp(-(\sqrt{2}\sigma_x)^2/2\sigma_x^2) + \sigma_x erf(\sqrt{2}\sigma_x/\sqrt{2}\,\sigma_x) < \sigma_x$$

After some simplification we have:

$$\sigma_x > \frac{\sqrt{2/\pi}\exp(-1)}{1 - erf(1) + 2/\sqrt{\pi}\exp(-1)} \approx 0.513$$

So, considering $\sigma_x > 0.513$ and $\delta = \sqrt{2}\sigma_x$ implies $\alpha' < \sigma_x$, and the proof is complete.□

Following section shows some experimental results demonstrating the better performance of the QX-VSS LMS against to some others.

IV. Experimental results: Predicting a noisy chirp signal

As mentioned earlier on theorem 1, the proposed filter weights converges to Wiener optimum weights. Also, we saw in Theorem 3, that the tracking performance of QX-VSS LMS algorithm is better than conventional LMS, based on some conditions. The result of running our algorithm for 40 noisy signals, regarding the theorem 3 conditions, shows that the proposed method can produces better results comparing to conventional LMS. Fig.2 shows the mean squared error (MSE) of the proposed QX-VSS LMS and conventional LMS predicted signals with the original noisy signal. The algorithms were run on 40 random chirp sinusoidal signals and for every signal the error estimation were computed. As can be seen the related MSE of the proposed filter is lower than conventional LMS.

Fig.3 shows the MSE of the LMS, VSS LMS and the proposed QX-VSS LMS weights. The results show that the proposed QX-VSS LMS algorithm has better tracking performance in comparison to VSS-LMS and conventional LMS algorithm.

V. Conclusion

Prediction is a major part of tracking algorithms. One of the most commonly used algorithms in prediction is the LMS algorithm. In this paper we proposed a new quantized input variable step size LMS, namely, the QX-VSS LMS algorithm. This algorithm uses a quantization scheme which involves the threshold clipping of the input signal. Mathematical analysis shows the convergence of the filter weights to the optimum Wiener weights. Also we showed under which circumstances tracking performance of QX-VSS LMS algorithm is better than conventional LMS. Experimental results on predicting a noisy chirp signal demonstrated the good performance of the proposed algorithm.

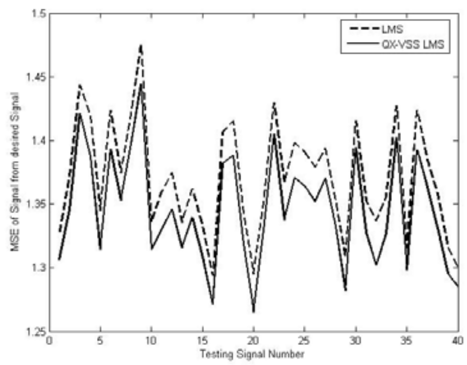

Fig. 2. Comparing MSE of the proposed QX-VSS LMS, conventional LMS and VSS LMS predicted signals with the original noisy signal.

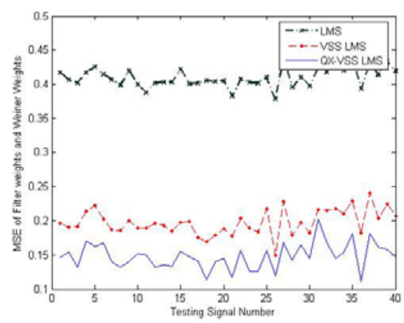

Fig. 3. Weight estimation error, MSE of the difference weight vector between proposed QX-VSS LMS, conventional LMS, VSS LMS weights and Wiener optimum weights.

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APPENDIX

Lemma 1. *If two random variables u and v both have a Gaussian distribution* $N(0,\sigma_u)$ *and* $N(0,\sigma_v)$ *respectively and* $E\{uv\}=\rho\sigma_u\sigma_v, \hat{v}=tgn(v,\sigma)$ *then:*

$$E\{u\hat{v}\}=\frac{\alpha'}{\sigma_v}E\{uv\}$$

Where $\alpha'=\sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2)+\sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$

Proof. We define the random variable $z=\frac{u}{\sigma_u}+\rho\frac{v}{\sigma_v}$

Now we have:

$$E\{zv\}=E\left\{\left(\frac{u}{\sigma_u}-\rho\frac{v}{\sigma_v}\right)v\right\}=E\left\{\frac{uv}{\sigma_u}\right\}-E\left\{\frac{\rho v^2}{\sigma_v}\right\}$$

With regard to assumption of the theorem

$$E\{zv\}=\frac{\rho\sigma_u\sigma_v}{\sigma_u}-\frac{\rho\sigma_v^2}{\sigma_v}=0$$

Therefore z and v are uncorrelated. Here we show that z and $\hat{v}$ are uncorrelated too. We have to show $E\{z\hat{v}\}=0$. At first we prove that $E\{\hat{v}\}=0$.

$$E\{\hat{v}\}=\int_{-\infty}^{\infty}\hat{v}f(\hat{v})d\hat{v}=\int_{-\infty}^{\infty}\hat{v}\exp(\frac{-\hat{v}^2}{\sigma_{\hat{v}}^2})d\hat{v}$$

$$=\int_{-\infty}^{-\delta}-\exp(\frac{-1}{\sigma_{\hat{v}}^2})d\hat{v}+\int_{-\delta}^{\delta}v\exp(\frac{-v^2}{\sigma_{\hat{v}}^2})dv+\int_{\delta}^{\infty}\exp(\frac{-1}{\sigma_{\hat{v}}^2})d\hat{v}=0$$

$$E\{z\hat{v}\}=E\{z\}E\{\hat{v}\}=0\Rightarrow E\left\{\left(\frac{u}{\sigma_u}-\rho\frac{v}{\sigma_v}\right)\hat{v}\right\}=0$$

$$\Rightarrow\frac{1}{\sigma_u}E\{u\hat{v}\}=\frac{\rho}{\sigma_v}E\{v\hat{v}\}$$

Therefore, we have:

$$E\{u\hat{v}\}=\rho\frac{\sigma_u}{\sigma_v}E\{v\hat{v}\} \qquad \text{(A-1)}$$

On the other hand

$$v\hat{v}=v\times tgn(v,\delta)=\begin{cases}|v|, & |v|>\delta\\ v^2, & |v|\le\delta\end{cases}$$

$$E\{v\hat{v}\}=\int_{-\infty}^{\infty}v\hat{v}\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{\sigma_v^2})dv=\int_{-\infty}^{-\delta}|v|\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_v^2})dv+$$

$$\int_{-\delta}^{\delta}v^2\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv+\int_{\delta}^{\infty}|v|\frac{1}{\sqrt{2\pi}\delta_v}\exp(\frac{-v^2}{2\sigma_v^2})dv$$

$$\Rightarrow E\{v\hat{v}\}=\frac{2}{\sqrt{2\pi}\delta_v}\int_{+\delta}^{+\infty}v\exp(\frac{-v^2}{2\sigma_v^2})dv+\frac{1}{\sqrt{2\pi}\delta_v}\int_{-\delta}^{+\delta}v^2\exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv$$

Here we calculate the above two terms:

$$\frac{2}{\sqrt{2\pi}\delta_v}\int_{+\delta}^{+\infty}v\exp(\frac{-v^2}{2\sigma_v^2})dv=\sqrt{\frac{2}{\pi}}\sigma_v\exp(\frac{-\delta^2}{2\sigma_v^2})$$

$$\frac{1}{\sqrt{2\pi}\delta_v}\int_{-\delta}^{+\delta}v^2\exp(\frac{-v^2}{2\sigma_{\hat{v}}^2})dv=\frac{1}{\sqrt{2\pi}\delta_v}\{-2\delta\exp(\frac{-\delta^2}{2\sigma_{\hat{v}}^2})\sigma_v^2$$

$$+\sigma_v^3 erf(\frac{\delta}{\sqrt{2}\sigma_v})\}=-\delta\sqrt{\frac{2}{\pi}}\sigma_v\exp(-\frac{\delta^2}{2\sigma_v^2})+\sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v})$$

Substituting the 2,3 in 1, and after some simplifications we have:

$$E\{v\hat{v}\}=\sqrt{\frac{2}{\pi}}\sigma_v(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2})+\sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v}) \qquad \text{(A-2)}$$

Now we can compute $E\{u\hat{v}\}$.Regarding (A-1) and (A-2) we have:

$$E\{u\hat{v}\}=\frac{\rho\sigma_u}{\sigma_v}\left\{\sqrt{\frac{2}{\pi}}\sigma_v(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2})+\sigma_v^2 erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\}$$

$$E\{u\hat{v}\}=\rho\sigma_u\sigma_v\left\{\frac{1}{\sigma_v}\sqrt{\frac{2}{\pi}}(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2})+erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\}$$

$$E\{u\hat{v}\}=\frac{1}{\sigma_v}E\{uv\}\left\{\sqrt{\frac{2}{\pi}}(1-\delta)\exp(\frac{-\delta^2}{2\sigma_v^2})+\sigma_v erf(\frac{\delta}{\sqrt{2}\sigma_v})\right\}$$

Hence with defining α' as follows:

$$\alpha'=\sqrt{2/\pi}(1-\delta)\exp(-\delta^2/2\sigma_v^2)+\sigma_v erf(\delta/\sqrt{2}\,\sigma_v)$$

So: $E\{u\hat{v}\}=\frac{\alpha'}{\sigma_v}E\{uv\}$. □