DESIGN OF LINEAR PHASE CIRCULARLY SYMMETRIC TWO-DIMENSIONAL FIR AND IIR DIGITAL FILTERS USING SCHUR DECOMPOSITION AND SYMMETRIES

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ABSTRACT

This paper presents a new and numerically efficient technique to design circularly symmetric two-dimensional (2-D) digital filters. This technique is based on two steps and in both we are using the Schur decomposition method (SDM): First, the 2-D impulse response matrix is decomposed into a parallel realization of K branches, each branch is composed of two cascaded SISO 1-D FIR digital filters. Second, a model reduction algorithm is applied to the 1-D filter to approximate the N-dimensional FIR into an n-dimensional IIR filters, where $n < N$. The model reduction algorithm is based on finding the eigenspaces associated with the large eigenvalues of the cross-Gramian matrix W_{CO}. It is shown that using the symmetry of the 2-D impulse response matrix and the fact that the left and right eigenspaces obtained by the SDM are transpose of each other, the design problem of *two* 1-D digital filters is reduced to the design problem of only *one* 1-D digital filter in each branch. Moreover, the symmetry property is exploited in the decomposition step, where we showed that we could apply the Schur decomposition to the leading principal minor of the impulse response matrix, and, by a simple manipulation we could find the decomposition of the whole impulse matrix. Thus, the computational effort is reduced substantially. A design example is given to illustrate the advantages of the proposed technique.

Keywords: *2-D Digital Filters, Circulary Symmetric, Schur Decomposition, Symmetry, Parallel Realization.*

الملخص

تقدم هذه الورقة طريقة جديدة ذات كفاءة عالية لتصميم المرشحات الرقمية ثنائية الأبعاد وذات التماثل الدائري باستخدام طريقة شور التجزئية . تعتمد هذه الطريقة على مرحلتين : الأولى تجزيء مصفوفة النبضات والتي تعرف بـ "Hankel Matrix" إلى عدد من الفروع المتوازية يمثل كل واحد منها الاستجابة النبضية لمرشحين رقميين متعاقبين

أحاديٌ الأبعاد أحدهما في الاتجاه الأفقي والآخر في الاتجاه الرأسي وهذه المرشحات هي مرشحات ذات استجابة نبضية محدودة (FIR) وفي الغالب تكون ذات رتبة عالية.

في الخطوة الثانية تبسط هذه المرشحات إلى مرشحات ذات استجابة نبضية لامحدودة **(IIR)** لكنها ذات رتبة أقل باستخدام ما يسمى " خوارزمية اختزال النموذج" مع السعي للاحتفاظ بخطية الطور للمرشح المصمم.

وقد أظهرت النتائج أن بالإمكان تصميم مرشح واحد في كل فرع من الفروع بدلاً من تصميم مرشحين كما في الطرق التقليدية. أيضا نظرا لوجود تماثل في مصفوفة هانكل، لم تطبق طريقة شورالتجزيئية إلا على جزء من مصفوفة هانكل وهذا بالتالي يقلل من العمليات الحسابية في التصميم بصورة كبيرة جداً.

في نهاية هذه الورقة أعطي مثال تصميمي لمرشح رقمي لبيان مزايا وكفاءة الطريقة المقترحة.

1. INTRODUCTION

Two-dimensional (2-D) digital filters are used in many applications such as image processing, seismic or geophysical signal processing, ultrasonic data processing and biomedical tomography. These applications might involve images for which the phase linearity is of great importance [Huang et al, 1975]. The design of linear phase two-dimensional (2-D) digital filters can be accomplished by using the window method [Huang, 1972 and Speake et al, 1981], the transformation of 1-D filters [Lien, 1992 and the references therein], and by using the singular value decomposition (SVD) (see [Hinamoto and Fairman, 1981, Kumar et al, 1987 and Lu et al, 1990 and 1991]. The later technique has received a considerable attention in the past few years. The reason for this interest is because, it offers, as pointed out in [Lu et al, 1990 & 1991], the following advantages. First, the design can be accomplished by designing a set of 1-D subfilters and, therefore, the well-established algorithms for the design of 1-D filters can be employed. Second, the stability issues of the 2-D filter is guaranteed if the 1-D subfilters employed are stable, and third, the 1-D subfilters form a parallel structure that allows extensive parallel processing.

The SVD can be applied to impulse response matrix (input-output data) as in [Kwan and Chan, 1989 and Lien, 1992], or it can be applied to the sampled magnitude response as in [Gu and Shenoi, 1991 and Lu et al, 1990 & 1991]. In [Hinamoto and Fairman, 1981, Kumar et al, 1987], a state space model of the separable-denominator transfer function is obtained, while in [Lin, et al, 1987], the state space representation is obtained by decomposing the 2-D impulse response matrix into two 1-D digital filters in cascade (single-input multi-output and multi-input single-output). Moreover, it was shown that an optimal decomposition could be obtained. In conjunction with the decomposition, in some of the previous work [Kumar et al, 1987, Lin, et al, 1987 and Lu, et al 1991], the balanced model reduction is employed to the decomposed state space representations to obtain computationally efficient filters.

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In this paper, we propose a new and computationally efficient algorithm to design linear phase 2-D digital filters using the Schur decomposition method (SDM). This method is reliable and numerically stable [Aldhaheri, 1991 and Laub, 1979] and it shares the SVD the forenamed advantages. Moreover, SDM enables us to exhibit the symmetries and utilize them in getting more computationally efficient filters design. The SDM is applied twice. Once, to decompose the impulse response matrix of the 2-D digital filters into horizontal and vertical 1-D sub-filters connected in cascade, and second, to approximate these subfilters, which can be considered as high order FIR filters, by reduced order IIR filters. The algorithm of the approximation is based on finding the orthonormal eigenspaces that correspond to the large eigenvalues of the cross-Gramian matrix W_{CO} of the 1-D FIR digital filter [Aldhaheri, 1997]. This algorithm avoids computing the balancing transformation, which tends to have numerical difficulties and ill-conditioning problem. The symmetry of this class of filters [Pei and Shyu, 1995] is utilized in the two steps to reduce the computational operations and to exhibit the possibility of having linear phase digital filters. In the decomposition step, we apply the SDM to the leading principal minor submatrix of the impulse response matrix; Moreover, it is shown that the Schur decomposition preserves some of the special structure of the 2-D impulse response matrix. That is, the coefficients of the transfer function of the decomposed 1-D filter are symmetric about their midpoints. This of course leads to having a linear phase FIR digital filter. In the approximation step, we showed that in each branch it is sufficient to design only one filter, either the one in the horizontal or the one in the vertical axis, i.e., in z_1 or z_2 domains, because they are identical. Furthermore, we showed through examples that the model reduction algorithm does not violate the linear phase property associated with the decomposition. A sign weight, (± 1) between the interconnected filters has to be introduced as it will be shown later in Section 3.

2. SCHUR DECOMPOSITION METHOD

The transfer function of 2-D digital filter is defined as

$$H(z_1,z_2)=\sum_{n_1=-N_1/2}^{N_1/2}\sum_{n_2=-N_2/2}^{N_2/2}h(n_1,n_2)\,z_1^{-n_1}z_2^{-n_2} \tag{1}$$

where $\mathrm{h}(\mathrm{n}_1,\mathrm{n}_2)$ is the impulse response of the filter. For the class of filters that we are considering here, *circularly symmetric* filter, the impulse response is real and it has the property: $h(n_1,n_2)=h(-n_1,-n_2)=h(-n_1,n_2)=h(n_1,-n_2)$. Moreover, it is diagonally symmetric, i.e., $h(n_1,n_2)=h(n_2,n_1)$. The 2-D transfer function $H(z_1,z_2)$ can be decomposed as:

$$H(z_1,z_2)=F(z_1)G(z_2)=\sum_{i=1}^{K}F_i(z_1)G_i(z_2) \tag{2}$$

where $F_i(z_1)$ and $G_i(z_2)$ are the transfer functions of the two cascaded SISO 1-D subfilters in z_1 and z_2 domains, respectively, and k is the number of parallel branches. As we will see later, the choice of k determines the allowable error in the frequency response of the designed filter. Let us assume that these subfilters are linear phase FIR filters. Then, the ith transfer function is given by:

$$F_i(z_1) = \sum_{n_1=0}^{N} f_i(n_1) z_1^{-n_1} \tag{3}$$

$$G_i(z_2) = \sum_{n_2=0}^{N} g_i(n_2) z_2^{-n_2} \tag{4}$$

Now, given the desired impulse response specifications $h_d(n_1,n_2)$ at the support $S_h = \{(n_1,n_2): 0 \le n_i \le N,\ i=1,2.\ N \text{ is assumed to be even}\}$. The desired impulse response in a matrix form is:

$$H_d = \begin{bmatrix} h_d(0,0) & h_d(0,1) & . & . & h_d(0,N) \\ h_d(1,0) & h_d(1,1) & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ h_d(N,0) & . & . & . & h_d(N,N) \end{bmatrix}, \quad \text{rank}(H_d) = r. \tag{5}$$

Then, the given H_d can be decompose into F and G of dimensions $(N+1)\times r$ and $r\times(N+1)$, respectively by using the Schur decomposition method (SDM), see the definition of SDM in [Aldhaheri, 1991 and Laub, 1979]. Before we perform the decomposition, let us first, rewrite the matrix H_d as follows:

$$H_d = \begin{bmatrix} I_L & 0 \\ \hat{I} & I_M \end{bmatrix} \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_L & \hat{I}^T \\ 0 & I_M \end{bmatrix} = \begin{bmatrix} H_1 & H_1\hat{I}^T \\ \hat{I}H_1 & \hat{I}H_1\hat{I}^T \end{bmatrix} \tag{6}$$

Where $H_1 \in R^{L\times L}$ is the leading principal minor of the matrix H_d, I_L and I_M are identity matrices of dimensions $L\times L$ and $M\times M$, respectively, and $\hat{I}$ is an $M\times L$ matrix and it is defined as: $\hat{I} = \begin{bmatrix} 0 & . & . & . & 1 & 0 \\ 0 & . & . & 1 & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{bmatrix}$, where $L = \frac{N}{2}+1$ and $M = \frac{N}{2}$.

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By close look at equation (6), it is clear that there are at most L linearly independent columns (or rows). This implies that

$$r = \text{rank}(H_d) \leq L \tag{7}$$

In general, the SDM decomposes the matrix into an upper quasi-triangular matrix and the eigenvalues of the matrix appear in a descending (or ascending) order of absolute value along the diagonal of the transformed matrix, but because of the symmetry of H_d, SMD yields:

$$V^T H_d V = \Lambda, \text{ where } \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots., \lambda_{N+1}) \tag{8}$$

where $V^T V = VV^T = I_{N+1}$. Now, partition the matrix V such that

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^T \begin{bmatrix} H_1 & H_1\hat{I}^T \\ \hat{I}H_1 & \hat{I}H_1\hat{I}^T \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots., \lambda_L)$. Notice here that, it is not necessary that all the eigenvalues of Λ_1 are nonzero or positive. In fact, the last $(L-r)$ eigenvalues of Λ_1 are zeros. The singular values $\sigma_i(H_d)$ are given by

$$\sigma_i = |\lambda_i|, \text{ for } i = 1, 2, \ldots\ldots\ldots, r$$

where $|.|$ denotes the absolute value of its elements.

Now, from equation (9) and the fact that the matrix V is orthonormal, we obtain the following:

$$V_{21} = \hat{I}V_{11}, \tag{10a}$$

$$V_{22} = -\hat{I}V_{12}, \tag{10b}$$

$$4V_{11}^T H_1 V_{11} = \Lambda_1, \tag{10c}$$

and

$$V_{11}^T V_{11} = 0.5 I_L \tag{10d}$$

Similarly, if we apply the SDM to the principal minor matrix H_1, we obtain

$$U^T H_1 U = \Sigma, \tag{11}$$

where $\Sigma = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots\ldots, \bar{\lambda}_L)$ and $U^T U = U U^T = I_L$. From (10) and (11) it can be shown that

$$V_{11} = \frac{1}{\sqrt{2}} U \tag{12a}$$

$$\Lambda_1 = 2\Sigma \tag{12b}$$

which implies that $\lambda_i = 2\bar{\lambda}_i$, for $i = 1, 2, \ldots\ldots, L$. Again, notice that the last $(L - r)$ eigenvalues of Σ are zeros. From equations (10) to (12) it is clear that it is sufficient to apply the SDM to the submatrix H_1, and from which we can find the decomposition of H_d. Therefore, H_d can be expressed as

$$H_d = V \Lambda V^T = \bar{V} \Lambda_1 \bar{V}^T = \bar{V}(2\Sigma)\bar{V}^T, \tag{13}$$

where

$$\bar{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} U \\ \hat{I} U \end{bmatrix} \tag{14}$$

The matrices $\bar{V} \in R^{(N+1)\times L}$ and $\bar{V}^T \in R^{L \times (N+1)}$ span the right and the left eigenspaces associated with Λ_1. Moreover, if the singular values of H_d, for $i > K$ is small, then H_d can be approximated as

$$H_d \approx V_1 (2\Sigma_1) V_1^T, \tag{15}$$

where V_1 is the first K columns of the matrix $\bar{V}$, $K < r$ and $\Sigma_1 = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots\ldots, \bar{\lambda}_K)$.

Substitute for V_1 in (15), yields

$$H_d \approx W |\Sigma_1| S W^T = FSG, \tag{16}$$

where

$$F = W \sqrt{|\Sigma_1|}, \tag{17a}$$

$$G = \sqrt{|\Sigma_1|}\, W^T = F^T, \tag{17b}$$

$$W = \begin{bmatrix} U(:,1:K) \\ \hat{I} U(:,1:K) \end{bmatrix}, \tag{17c}$$

And $S = diag(s_1, s_2, \ldots\ldots, s_K) = \mathrm{diag}(\mathrm{sign}(\bar{\lambda}_1), \mathrm{sign}(\bar{\lambda}_2), \ldots\ldots, \mathrm{sign}(\bar{\lambda}_K))$. (17d)

$U(:,1:K)$ denotes a submatrix of U, whose rows equal to U and columns are the first K columns of U. Notice here that the decomposed matrix F (and G) has a nice property, as H_d, that is, each column of F (and row of G) is symmetric about its midpoint. Now, if we consider these columns of F (or rows of G) characterizing SISO 1-D subfilters, then each one will be consider as a linear phase FIR filter.

3. LINEAR PHASE 2-D IIR DIGITAL FILTER DESIGN

In this section, we also use the Schur decomposition method to convert the decomposed SISO FIR digital filter of order N to corresponding IIR filter approximation of order n with the aim of preserving the phase linearity. In order to avoid the ill-conditioning problem, which is usually associated with computing the balanced transformation matrix, we propose an algorithm which depends on finding the left and right eigenspaces of the large eigenvalues of the cross-Gramian matrix W_{CO} of the 1-D FIR digital filter. The reduced order IIR obtained by this algorithm is input/output equivalent to the balanced IIR filter, but it is obtained with out any matrix inversion. The phase linearity is preserved in the most concerned band, the passband. Furthermore, since the 2-D filter consists of parallel branches of these IIR filters, the resulting 2-D digital filter is IIR with linear phase in the passband.
We start our model reduction algorithm by substituting (17) into (16) and rewriting equation (16) as:

$$H_d \approx \sum_{i=1}^{K} F_i s_i G_i \tag{18}$$

Where F_i is the ith column of the matrix F, and $G_i = F_i^T$ is the ith row of the matrix G and $s_i = \text{sign}(\overline{\lambda}_i) = \pm 1$. Now, the ith 1-D FIR filters are characterized by

$$F_i(z_1) = \sum_{j=0}^{N} f_{ji}\, z_1^{-j} \tag{19a}$$

and

$$G_i(z_2) = \sum_{j=0}^{N} g_{ji}\, z_2^{-j} \tag{19b}$$

Since $G_i = F_i^T$, it is enough to design only *one* filter in each branch of Fig.1.

From (18) and (19), the transfer function of the linear phase 2-D FIR filter is characterized by

$$H_{FIR}(z_1, z_2) = \sum_{i=1}^{K} F_i(z_1)\; s_i\; F_i(z_2)$$

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The parallel realization of this filter is shown in Fig. 1.

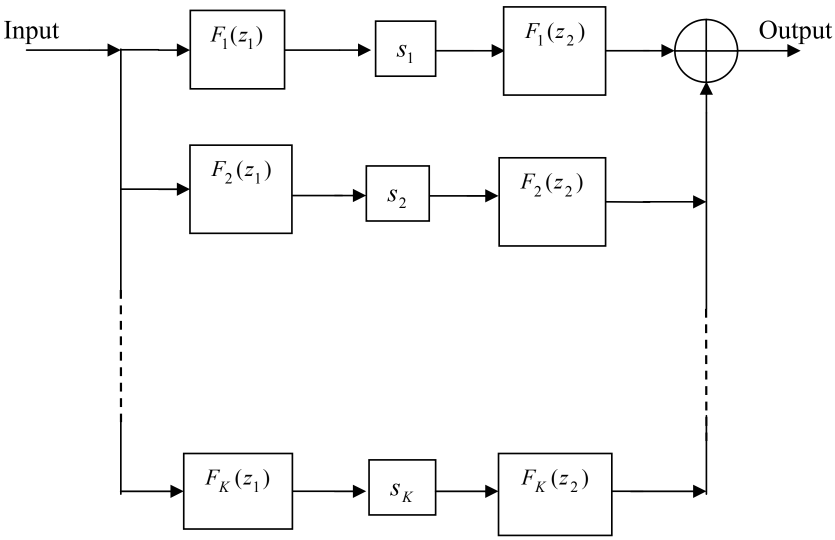

Fig. 1. Parallel realization of 2-D FIR digital filter based on SDM and symmetries.

For the sake of simplicity, let us drop the subscripts of F and z and rewrite equation (19a) as:

$$F_i(z_1) = H(z) = \sum_{j=0}^{N} c_j z^{-j} = C\left[zI_N - A\right]^{-1} B + D \qquad (20)$$

As we mentioned before, the filter characterized by equation (20) is a linear phase FIR digital filter. Therefore,

$$c_j = c_{N-j}, \quad \text{for } j = 0, 1, \ldots\ldots\ldots, N \qquad (21)$$

The controllable canonical state space (A_c, B_c, C_c, D_c) of this filter takes the form

$$A_c = \begin{bmatrix} 0 & 0 & . & . & 0 \\ 1 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . \\ . & . & . & . & . \\ 0 & . & . & 1 & 0 \end{bmatrix},\ B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, C_c = [c_1\ c_2\ \ldots\ldots\ c_N] \text{ and } D_c = c_0. \qquad (22)$$

The purpose of the model reduction is to convert the N^{th} order FIR, characterized by (20) to an n^{th} order IIR digital filter, where $n < N$ through a similarity transformation T.

Define the cross-Gramian matrix W_{CO} as

$$W_{CO} = \sum_{k=0}^{\infty} A_c{}^k B_c C_c A_c{}^k \tag{23}$$

Equivalently, W_{CO} can be computed by solving the Lyapunov equation

$$A_c W_{CO} A_c - W_{CO} + B_c C_c = 0 \tag{24}$$

Notice that W_{CO} is invariant under the similarity transformation [Aldhaheri, 1991] and the singular values $\hat{\sigma}_i$, are given by

$$\hat{\sigma}_i(H(z)) = \left|\lambda_i(W_{CO})\right|, \quad i = 1, 2, , ..., N \tag{25}$$

The order n is chosen based on a specified error bound, ε which satisfies the inequality

$$\left\|\mathrm{H(z) - H_n(z)}\right\|_\infty \leq \varepsilon = 2\sum_{\mathrm{i=n+1}}^{\mathrm{N}} \hat{\sigma}_{\mathrm{i}} \tag{26}$$

where $\|\ \|_\infty$ denotes the maximum absolute value of its frequency response and $\mathrm{H_n(z)}$ is the transfer function of the reduced order IIR filter and defined as

$$H_n(z) = D_n + C_n[zI_n - A_n]^{-1} B_n. \tag{27}$$

The state space representation of the n^{th} order reduced order IIR filter (A_n, B_n, C_n, D_n) will be defined later in this section.

Because of the special structure of the state space of the FIR filter (22), equation (23) is simplified to:

$$W_{co} = \sum_{k=0}^{N-1} A_c{}^k B_c C_c A_c{}^k = \Omega_C \Omega_O, \tag{28}$$

where Ω_C and Ω_O are the controllability and the observability matrices, respectively, which are defined as:

$$\Omega_C = \left[B_c \quad \mathrm{A_c B_c} \quad .... \quad \mathrm{A_c^{N\text{-}1}} B_c\right] \text{and } \Omega_{\mathrm{O}} = \left[\mathrm{C_c^T} \quad \mathrm{A_c^T C_c^T} \quad ...... \quad (\mathrm{A_c^T})^{\mathrm{N\text{-}1}} C_c^T\right]^T \tag{29}$$

Notice also that A_c is nilpotent since $A_c^N = 0$. Therefore, equation (28) yields

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$$W_{CO} = \begin{bmatrix} c_1 & c_2 & . & . & c_N \\ c_2 & c_3 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ c_N & 0 & . & . & 0 \end{bmatrix} \tag{30}$$

Again, notice that the matrix W_{CO} is symmetric and is easily constructed. Therefore, there is no computation involved in finding W_{CO}.

In the rest of this section, we summarize briefly, the model reduction algorithm in the following steps:

i. For each branch, and from (19a) and (20), find the impulse response,

$c_i, i = 0, 1, \ldots\ldots\ldots\ldots, N$, Characterizing the linear phase FIR filter. From this construct the state-space representation (A_c, B_c, C_c, D_c) as in (22).

ii. Construct W_{co} as in equation (30)

iii. Compute the Schur decomposition of W_{CO},

$$T^T W_{CO} T = \Delta \tag{31}$$

where $\Delta = \text{diag}(\delta_1, \delta_2, \ldots\ldots\ldots\ldots, \delta_N)$ with the ordering

$$|\delta_1| \geq |\delta_2| \geq \ldots\ldots\ldots \geq |\delta_n| \geq |\delta_{n+1}| \geq \ldots\ldots\ldots \geq |\delta_N|$$

The desired order of the IIR filter is determined based on the required error bound ε, which is defined by equation (26), where $\hat{\sigma}_i = |\delta_i|$ for all i.

iv. Partition the matrices T and T^T such that

$$\begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix} W_{CO} [T_1 \quad T_2] = [\Delta_1 \quad \Delta_2], \tag{32}$$

where $\Delta_1 = \text{diag}(\delta_1, \delta_2, \ldots\ldots\ldots\ldots, \delta_n)$ and $\Delta_2 = \text{diag}(\delta_{n+1}, \delta_{n+2}, \ldots\ldots\ldots\ldots, \delta_N)$.

The matrices $T_1 \in R^{N\times n}$ and T_1^T span the right and the left eigenspaces associated with Δ_1 Similarly, $T_2 \in \mathrm{R}^{\mathrm{N\times N\text{-}n}}$ and T_2^T span the right and left eigenspaces associated with Δ_2.

v. Apply this transformation, T to (A_c, B_c, C_c, D_c) to obtain

$$\begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix} A_c \, [T_1 \quad T_2] = T^T(:,2:N)T(1:N-1,:) = \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} \tag{33a}$$

$$\begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix} B_c = T^T(:,1) = \widetilde{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{bmatrix}, \quad C_c \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{bmatrix} \text{and } \widetilde{D} = D_c \tag{33b}$$

vi. The state space of the n[th] order-reduced model (A_n, B_n, C_n, D_n) is defined as:

$$A_n = \widetilde{A}_{11} = \mathrm{T}^{\mathrm{T}}(1:n,2:N)T(1:N-1,1:n) \tag{34a}$$

$$B_n = \widetilde{B}_1 = T^T(1:n,1) \tag{34b}$$

$$C_n = \widetilde{C}_1 = CT(:,1:n) \tag{34c}$$

$$D_n = \widetilde{\mathrm{D}} = D_c, \tag{34d}$$

where $T(i:j,k:l)$ denotes an extraction of the rows of the matrix T from i to j and columns from k to l. Thus, the reduced order IIR filter characterized by equation (27) is an approximation to the full order FIR filter characterized by equation (22).

Thus, the design of 2-D digital filter can be accomplished through the following steps;

1. Decompose the desired 2-D impulse response matrix, H_d by using the procedure described in Section 2. From this, determine the number of branches K and find the matrices F and S.
2. Design 1-D FIR filters characterized by equation (19). As we mentioned earlier, $G_i(z_2) = F_i(z_2)$. Therefore, the design problem is reduced to a design of a single 1-D digital filter.
3. Apply the model reduction algorithm described above to approximate the FIR by IIR digital filters.
4. Replace $F_i(z_j)$ for $i = 1,2,......,K$ and $j = 1,2$ by $H_{n_i}(z_j)$ in Fig.1, where n_i denotes the order of the i^{th} branch, to obtain the parallel realization of the linear phase 2-D IIR.
5. Insert the sign weight, (± 1) between the cascaded $H_{n_i}(z_1)$ and $H_{n_i}(z_2)$. The overall 2-D IIR digital filter is characterized by the transfer function:

$$H_a(z_1, z_2) = \sum_{i=1}^{K} H_{n_i}(z_1)\, s_i \, H_{n_i}(z_2) \tag{35}$$

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4. DESIGN EXAMPLE

In this section, a 2-D circularly symmetric low pass digital filter is designed to illustrate the effectiveness of the proposed technique in the decomposition and in the model reduction steps.

Example 1.

Consider the 2-D circularly symmetric low pass digital filter that satisfies the following:

$$|H_d(\omega_1,\omega_2| = \begin{cases} 1 & \text{for } 0 \le \sqrt{\omega_1^2+\omega_2^2} \le \omega_p \\ 0 & \text{for } \omega_s \le \sqrt{\omega_1^2+\omega_2^2} \le \pi \end{cases}$$

where $\omega_p = 0.4\pi$ and $\omega_s = 0.5\pi$. Thus, $\omega_c = 0.45\pi$, and the corresponding impulse response smeared by Kaiser window is given by

$$h_d(n_1,n_2) = \frac{0.225\, J_1(0.45\pi\sqrt{(n_1-20)^2+(n_2-20)^2}\,)}{\sqrt{(n_1-20)^2+(n_2-20)^2}} W_K(n_1-20, n_2-20), 0 \le n_1, n_2 \le 40$$

where $W_K(i,j) = \dfrac{I_0\left\lfloor \alpha\sqrt{1-(i^2+j^2)/400} \right\rfloor}{I_0(\alpha)}$, $I_0(x)$ is the zeroth order modified Bessel function of the first kind and α is a parameter determined by the allowable error in the passband and stopband. Following the procedure of Section 2, we applied the SDM to the first 21×21 principal submatrix of the 41×41 impulse response matrix. The rank of the impulse response matrix is $r = 21$, which means that the exact number of the parallel sections is 21. But if we neglect the singular values of less than 1% of the largest one, we come up with K equal to 5 and $S = diag(1,-1,1,-1,1)$. The resulting magnitude response of the 2-D FIR filter is shown in Fig. 2. The maximum error in the passband is 0.0128.

To check the linearity of the phase, the group delay of the 2-D FIR digital filter, which is defined as:

$$\tau_1(\omega_1,\omega_2) = -\frac{\partial\{\arg\lfloor H_{FIR}(\omega_1,\omega_2)\rfloor\}}{\partial\omega_1}$$

and

$$\tau_2(\omega_1,\omega_2) = -\frac{\partial\{\arg\lfloor H_{FIR}(\omega_1,\omega_2)\rfloor\}}{\partial\omega_2}$$

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is computed, where $H_{FIR}(\omega_1, \omega_2)$ is the frequency response of the 2-D FIR digital filter. It is found that the group delay is constant over the entire band. Similar result would be obtained if we compute the group delay $\tau_2(\omega_1, \omega_2)$.

The next step is to apply the proposed model reduction algorithm, described in Section 3, to these five 1-D FIR filters of order 40. Now, if we follow the steps of the algorithm proposed in Section 3, we will find that the IIR filters of order $13, 15, 16, 17, \text{and} \, 17$ give a satisfactory result. It is worth mentioning that the overall accuracy of the 2-D digital filter depends on the number of the branches, K and the approximation of the FIR by IIR filters, i.e., the higher the order we choose, the better results we achieve. This choice, of number of parallel sections and filters orders, gives a maximum error of 0.0132. The magnitude response of the approximated 2-D IIR digital filter is depicted in Fig. 3.

Fig. 4 illustrates the phase of the reduced order 2-D IIR filter in the normalized passband region. As seen from this figure, the phase is almost linear over the passband region. So, we can say that the linear phase is preserved over the passband region.

5. CONCLUSION

In this paper, a new design technique for passband linear phase circularly symmetric 2-D IIR digital filters using Schur decomposition is presented. This technique is based on two steps: decomposing the 2-D impulse matrix, H_d into K parallel sections of two 1-D FIR digital filters, and, the FIR filters of order N are converted to a reduced order IIR filters of order n_i, where $i = 1, 2, ..., K$ using the model reduction algorithm outlined in Section 3.

The symmetry of this type of 2-D filters is utilized in the two steps to reduce the computational operations. In the first step, the decomposition is applied to only $L \times L$ submatrix of H_d. Then the symmetry is employed to find the decomposition of the whole matrix, H_d. In the second step, since one of the two 1-D impulse response specification is the transpose of the other, it is sufficient to design one of them. Therefore, in the model reduction stage, the algorithm is applied once. Hence, the computational effort will be reduced substantially. The given examples have shown that the 2-D IIR digital filter magnitude response is very close to the ideal and the maximum error in the passband is very small. More over, the phase linearity in the passband is preserved.

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REFERENCES

1. Aldhaheri, R.W., 1991, "Model order reduction via real Schur form decomposition", *Int. J. Contr*., 53, (3), pp. 709-716.
2. Aldhaheri, R.W., and Al-Hussaini, N.B., 1997, "Approximation of FIR by IIR digital filters using orthonormal bases", *Proc. Internat. Conf. On Information, Communication and Sig. Process*., Singapore, 3, pp. 1311-1314.
3. Gu, G., and Shenoi, B.A., 1991, "A novel approach to the synthesis of recursive digital filters with linear phase", *IEEE Trans. Circuits Syst*., CAS-38, (6), pp. 602-612.
4. Hinamoto, T., and Fairman, F.W., 1981, "Separable-denominator state-space realization of two-dimensional filters using a canonic form', *IEEE Trans. Acoust. Speech Signal Process*., ASSP-29, (8), pp. 846-853.
5. Huang, T.S., et al,1975, "The importance of the phase in image processing filters", *IEEE Trans. Acoust. Speech Signal Process*. ASSP-23, (12), pp. 529-542.
6. Huang, T.S., 1972, "Two-dimensional windows", *IEEE Trans. Audio Electroacoust.,*Au-20, pp. 88-89.
7. Kumar, A., et al, 1987, "Separately balanced realization and model reduction of 2-D Separable-denominator transfer function from input-output data", *IEEE Trans. Circuits Syst.,* CAS-34, (3), pp. 233-239.
8. Kwan, H.K., and Chan, CL., 1989, "Design of linear phase circularly symmetric two-dimensional recursive digital filters', *IEEE Trans. Circuits Syst.,* CAS-36, (7), pp. 1023-1029.
9. Laub, A.J., 1979, "A Schur method for solving algebraic Riccati equations", *IEEE Trans. Autom. Control*, AC-24, (6), pp. 913-921.
10. Lien, B.K., 1992, "On the cascade realization of 2-D FIR filters designed by McClellan Transformation", *IEEE Trans. Signal Process.*, 40, (9), pp. 2338-2340.
11. Lin, T., et al, 1987, "Design of two-dimensional separable denominator digital filters based on the reduced-dimensional decomposition," *IEEE Trans. Circuits Syst.,* CAS-34, (8), pp. 934-941.
12. Lu, W-S., et al, 1990, "Design of two-dimensional FIR digital filters by using the singular value decomposition", *IEEE Trans. Circuits Syst*., CAS-37, (1), pp. 35-46.
13. Lu, W-S., et al, 1991, "Design of two-dimensional digital filters using the singular-value decomposition and balanced approximation method", *IEEE Trans. Signal Process*., 39, (10), pp. 2253-2262.
14. McClellan, J.H., 1973, "The design of two-dimensional filters by transformation," *Proc. 7th annual Princeton Conf. Information Sciences and Systems*, pp. 247-251.
15. Meeklenbrauker, W.F.G., and Mersereau, R.M., 1976, "McClellan transformation for two-dimensional digital filters II: Implementation", *IEEE Trans. Circuits Syst.,* CAS-23, (7), pp. 414-422.
16. Mersereau, R.M., et al, 1976, "McClellan transformation for two-dimensional digital filters: I-Design", *IEEE Trans. Circuits Syst.,* CAS-23, (7), pp. 405-413.
17. Pei, S-C., and Shyu, J-J., 1995, "Symmetric Properties of 2-D sequences and their applications for designing linear-phase 2-D FIR digital filters", *Signal Processing*, 42, (3), pp. 261-271.
18. Speake, T.C., and Mersereau, R.M., 1981, "A note on the use of windows for two-dimensional FIR filter design", *IEEE Trans. Acoust. Speech Signal Process.* ASSP-29, (2), pp. 125-127.

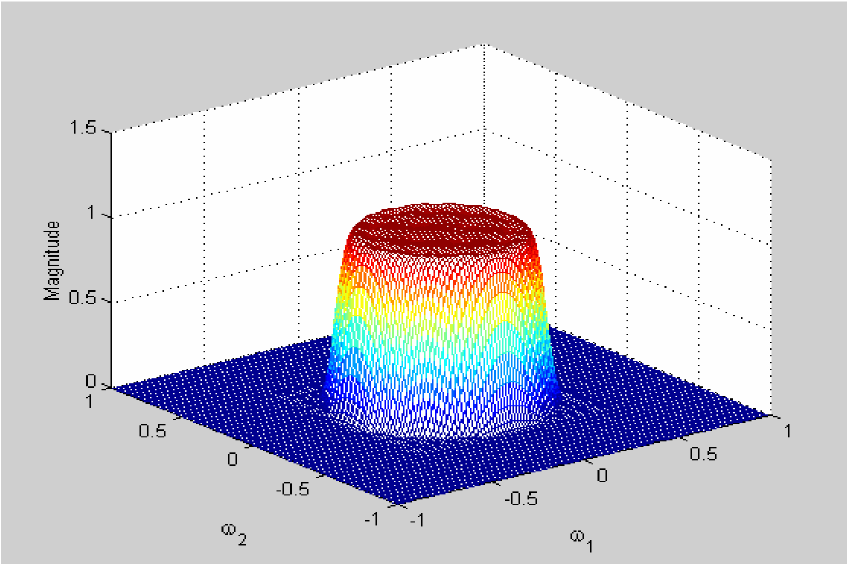

Fig. 2. Magnitude response of circularly-symmetric 2-D FIR low pass digital filter.

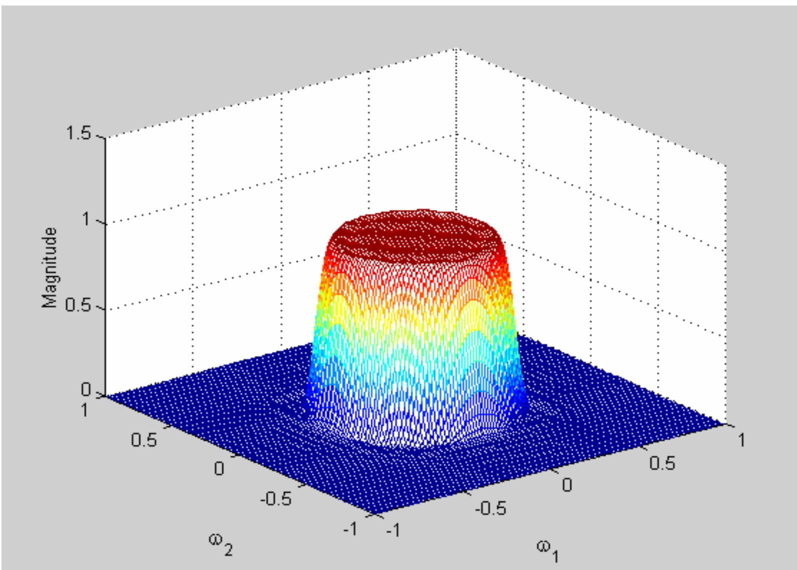

Fig. 3. Magnitude response of circularly symmetric 2-D IIR low pass filter using SDM and model reduction for $K = 5$ and $n_i = 13, 15, 16, 17$, and 17, for $i = 1, 2, \ldots, 5$.

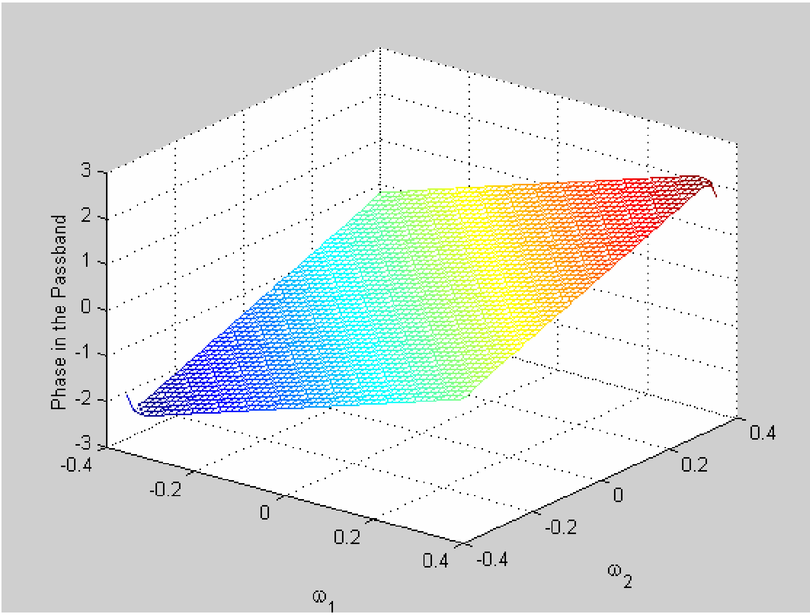

Fig. 4. Phase in the passband for the 2-D IIR low pass filter.

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