1. Let \( f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \) be a function. Define \( f(1) = (1, 1) \); and if \( f(k) = (a, b) \) define
\[
f(k + 1) = \begin{cases} 
(a + 1, b - 1) & \text{if } b > 1 \\
(1, a + 1) & \text{if } b = 1 
\end{cases}
\]
Thus, \( f(2) = (1, 2) \), \( f(3) = (2, 1) \) and so on.

(a) Prove that \( f \) is surjective.
(b) Prove that \( f \) is also injective.

**SOLUTION:** The function tells us that \( f(k + 1) \) follows \( f(k) \) as we move down \( f(k) \)'s diagonal, unless \( f(k) \) is already at the bottom, in which case \( f(k + 1) \) is at the top of the next diagonal.

(a) We will prove that \( f \) is surjective (onto) by contradiction. If \( f \) is NOT surjective, move along the diagonals in succession until the first \((a, b)\) is reached that it is not in range of \( f \). If \( a \neq 1 \) then \((a - 1, b + 1)\) precedes \((a, b)\) on the same diagonal, so \((a - 1, b + 1) = f(k) \) for some \( k \in \mathbb{Z} \). But then an application of the definition of \( f \) above gives \( f(k + 1) = (a, b) \), contradicting the assumption that \((a, b) \notin \mathbb{Z} \). If \( a = 1 \) then \((b - 1, 1)\) is at the bottom of the preceding diagonal, so \((b - 1, 1) = f(k) \) for some \( k \), and then the definition of \( f \) yields \( f(k + 1) = (1, b) = (a, b) \), again a contradiction. Thus \( f \) is injective.

(b) To prove that \( f \) is injective (one-to-one), we have to examine that
\[ f(k_1) = f(k_2) = (x, y) \rightarrow k_1 = k_2. \]
In other word, for one pair \((x, y)\) there is only one unique \( k \). By some arithmetic calculation, the relation between \( k \) and \((x, y)\) is:
\[
k = \frac{(x + y)^2 - x - 3y}{2} + 1, x \geq 1, y \geq 1
\]
It is easy also to check that \( k \geq 1 \). Suppose there are two different pairs \((x, y)\) and \((s, t)\) that produce the same \( k \). Then having substituted \( x, y, s, t \) into the equation, it yields:
\[
(x - s)(x + s) + (y - t)(y + t) + 2(xy - st) + (s - x) + 3(t - y) = 0.
\]
Because \(x, y, s, t \geq 1\), the only solution is \(x = s\) and \(y = t\). It means that \((x, y) = (s, t)\). This is a contradiction. Thus, we proved that for one pair \((x, y)\) there is only one unique \(k\).

2. Let \(f, g: \mathbb{R} \to \mathbb{R}\) be given by

\[
f(x) = \begin{cases} 
1 - 2x, & \text{if } x \geq 0 \\
| x |, & \text{if } x < 0
\end{cases}
\]

\[
g(x) = \begin{cases} 
3x, & \text{if } x \geq 0 \\
x - 1, & \text{if } x < 0
\end{cases}
\]

Find \(f \circ g\) and \(g \circ f\).

**SOLUTION:**

\[(f \circ g)(x) = f(g(x)):\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(g(x))</th>
<th>(f(g(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x \geq 0)</td>
<td>(3x \geq 0)</td>
<td>(1 - 2(3x))</td>
</tr>
<tr>
<td>(x &lt; 0)</td>
<td>(x - 1 &lt; -1)</td>
<td>(</td>
</tr>
</tbody>
</table>

Thus,

\[(f \circ g)(x) = \begin{cases} 
1 - 2(3x), & \text{if } x \geq 0 \\
|x - 1|, & \text{if } x < 0
\end{cases}
\]

\[(g \circ f)(x) = g(f(x)):\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(g(f(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 \leq x \leq 0.5)</td>
<td>(1 - 2x \geq 0)</td>
<td>(3(1 - 2x))</td>
</tr>
<tr>
<td>(x &gt; 0.5)</td>
<td>(1 - 2x &lt; 0)</td>
<td>((1 - 2x) - 1)</td>
</tr>
<tr>
<td>(x &lt; 0)</td>
<td>(</td>
<td>x</td>
</tr>
</tbody>
</table>

Thus,

\[(g \circ f)(x) = \begin{cases} 
-2x, & \text{if } x > 0.5 \\
3(1 - 2x), & \text{if } 0 \leq x \leq 0.5 \\
3 \cdot | x |, & \text{if } x < 0.
\end{cases}
\]

3. Let \(S\) and \(T\) be sets with three elements and two elements, respectively. In each case state the answer and justify briefly.

(a) How many functions are there from \(S\) to \(T\)?
(b) How many injections are there from \(S\) to \(T\)?
(c) How many surjections are there from \(S\) to \(T\)?
(d) How many bijections are there from \(S\) to \(T\)?

**SOLUTION:**

(a) Eight.
(b) Zero.
(c) Six.
(d) Zero.

4. Suppose \(f : \mathbb{N} \to A\) and \(g : \mathbb{N} \to B\) are surjections. Prove that there is a surjection \(h : \mathbb{N} \to A \cup B\).
5. Prove that 

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n + m - 1}{m} \right\rfloor,$$

for all integers $n$ and all positive integers $m$.

**SOLUTION:**
Suppose that $n = am + b$, where $a, b \in \mathbb{Z}, 0 \leq b < m$. Thus,

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{am + b}{m} \right\rfloor = \left\lfloor a + \frac{b}{m} \right\rfloor = \left\{ \begin{array}{ll} a & , b = 0 \\ a + 1 & , \text{otherwise} \end{array} \right.$$

Thus,

$$\left\lfloor \frac{n + m - 1}{m} \right\rfloor = \left\lfloor (a + 1) + \frac{b - 1}{m} \right\rfloor = \left\{ \begin{array}{ll} a & , b = 0 \\ a + 1 & , \text{otherwise} \end{array} \right.$$

6. Let $f(x) = \frac{(x + A)^2}{x^2 + 1}$, $-\infty < x < +\infty$, where $A$ is a positive constant. Show that $0 \leq f(x) \leq A^2 + 1$ for all $x \in \mathbb{R}$.

**SOLUTION:**
Let $y = f(x)$. Clearly $y \geq 0$. Also $y = \frac{(x + A)^2}{x^2 + 1} \iff (y - 1)x^2 - 2Ax + y - A^2 = 0$. This is a quadratic equation in $x$ and $x$ is real so the discriminant $D = b^2 - 4ac \geq 0$. But,

$$D = 4A^2 - 4(y - 1)(y - A^2) = 4[A^2 - (y^2 - y - A^2y + A^2)] = -4y(y - 1 - A^2)$$

Since $y \geq 0$ so $D \geq 0 \iff y - 1 - A^2 \leq 0 \iff y \leq 1 + A^2$. Hence, $0 \leq f(x) \leq A^2 + 1$.

7. Find the domain of each of the following functions:
   (a) $f(x) = \log(\log_{1/2}(x^2 + 4x + 3)) + \sin^{-1}[2x^2 - 3]$.
   (b) $g(x) = h(9x^2 - 1)$, where $h$ is a function defined on $[0, 2]$.
   (c) $k(x) = \log(ax^2 + bx + c)$.

**SOLUTION:**
(a) $f(x)$ is defined only if $\log_{1/2}(x^2 + 4x + 3) > 0$ and $[2x^2 - 3] \in [-1, 1]$. Clearly, $[2x^2 - 3]$ is either $-1, 0, 1$. Then, $-1 \leq 2x^2 - 3 < 2$. Thus, $\sin^{-1}[2x^2 - 3]$ is defined only if $x \in [1, \sqrt{5/2}) \cup (-\sqrt{5/2}, -1]$. Also $\log_{1/2}(x^2 + 4x + 3) > 0$ only if $0 < x^2 + 4x + 3 < 1 \iff -2 - \sqrt{2} < x < \sqrt{2} - 2$. Hence domain of $f(x)$ is $(-\sqrt{5/2}, -1) \cap (-2, \sqrt{2}) = (-\sqrt{5/2}, -1)$.

(b) $g(x)$ is meaningful if $0 \leq 9x^2 - 1 \leq 2 \iff 1 \leq 9x^2 \leq 3$. In other words, there are two inequalities must be fulfilled: $9x^2 - 1 \geq 0$ and $9x^2 - 3 \leq 0$. Thus, $x \in ([-\infty, -1/3] \cup (1/3, \infty]) \cap [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$.
(c) \( k(x) = \log(ax^2 + bx + c) \). There are six cases: If \( b^2 - 4ac > 0 \) and \( a > 0 \) then the function is defined on \( \mathbb{R} - [x_1, x_2] \), where \( x_1, x_2 \) are the roots of \( ax^2 + bx + c = 0 \). If \( b^2 - 4ac > 0 \) and \( a < 0 \) then the function is defined only on \( (x_1, x_2) \). If \( b^2 - 4ac < 0 \) and \( a > 0 \) then \( k \) is defined on \( \mathbb{R} \). If \( b^2 - 4ac < 0 \) and \( a < 0 \) then \( k \) is defined nowhere on \( \mathbb{R} \). If \( b^2 - 4ac = 0 \) and \( a > 0 \) then \( k \) is defined on \( \mathbb{R} - \{ -b/(2a) \} \). If \( b^2 - 4ac = 0 \) and \( a < 0 \) then \( k \) is defined nowhere on \( \mathbb{R} \).

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**SOLUTION:**

(a) TRUE. Since \( \lfloor \lceil x \rceil \rfloor = \lceil x \rceil \).

(b) FALSE. A counterexample: \( x = y = \frac{1}{2} \).

(c) TRUE. Since we are dividing by 4, let \( x = 4n + k \), where \( 0 \leq k < 4 \). There are 3 cases. CASE 1, \( k = 0 \): If \( k = 0 \), then \( x \) is already a multiple of 4, so both sides equal \( n \). CASE 2, \( 0 < k \leq 2 \): then \( \lfloor x/2 \rfloor = 2n + 1 \), so the left side is \( \lfloor n + \frac{1}{2} \rfloor = n + 1 \). Of course the right side is \( n + 1 \) as well, so again the two sides agree. CASE 3, \( 2 < k < 4 \): Then \( \lfloor x/2 \rfloor = 2n + 2 \), and the left side is \( \lfloor n + 1 \rfloor = n + 1 \); of course the right side is still \( n + 1 \), as well.

(d) FALSE. A counterexample: \( x = 8.5 \).

(e) TRUE. Write \( x = n + \epsilon \) and \( y = m + \sigma \), where \( n \) and \( m \) are integers and \( \epsilon \) and \( \sigma \) are nonnegative real number less than 1. The left side is \( n + m + (n + m) \) or \( n + m + (n + m + 1) \), the latter occurring if and only if \( \epsilon + \sigma \geq 1 \). The right side is the sum of two quantities. The first is either \( 2n \) (if \( \epsilon < \frac{1}{2} \) or \( 2n + 1 \) (if \( \epsilon \geq \frac{1}{2} \)). The second is either \( 2m \) (if \( \sigma < \frac{1}{2} \)) or \( 2m + 1 \) (if \( \sigma \geq \frac{1}{2} \)). The only way then for the left side to exceed the right side is to have the left side be \( 2n + 2m + 1 \) and the right side be \( 2n + 2m \). This can occur only if \( \epsilon + \sigma \geq 1 \) while \( \epsilon < \frac{1}{2} \) and \( \sigma < \frac{1}{2} \). But that is impossible, since the sum of two numbers less than \( \frac{1}{2} \) cannot be as large as 1. Therefore the right side is always at least as large as the left side.