1. Set Operations

2. Set Identities

3. Generalized Unions and Intersections

4. Computer Representation of Sets

5. The cardinality of the union of sets

6. Preview: Functions
Set Operations

Definition 1 Let \( A \) and \( B \) be sets. The union of the sets \( A \) and \( B \), denoted by \( A \cup B \), is the set that contains those elements that are either in \( A \) or in \( B \), or in both. That is,
\[
A \cup B = \{ x | x \in A \lor x \in B \}.
\]

Definition 2 Let \( A \) and \( B \) be sets. The intersection of the sets \( A \) and \( B \), denoted by \( A \cap B \), is the set that contains those elements in both \( A \) and \( B \). That is,
\[
A \cap B = \{ x | x \in A \land x \in B \}.
\]

Definition 3 Two sets are called disjoint if their intersection is \( \emptyset \), the empty set.

Definition 4 Let \( A \) and \( B \) be sets. The difference of the sets \( A \) and \( B \), denoted by \( A - B \), is the set that contains those elements that are in \( A \) but not in \( B \). The difference of \( A \) and \( B \) is also called the complement of \( B \) with respect to \( A \). That is,
\[
A - B = \{ x | x \in A \land x \notin B \} = A \cap \bar{B}.
\]
Definition 5 Let $A$ and $B$ be sets. The symmetric difference of the sets $A$ and $B$, denoted by $A \oplus B$, is the set that contains those elements any of the two sets but not in both. That is,

$$A \oplus B = A \cup B - A \cap B = (A - B) \cup (B - A).$$

Definition 6 Let $U$ be the universal set. The (absolute) complement of the set $A$, denoted by $\bar{A}$, is the complement of $A$ with respect to $U$. Or, the complement of the set $A$ is $U - A$. That is,

$$\bar{A} = \{x|\neg(x \in A)\} = \{x \in U|x \notin A\}.$$ 

Example: Let

$$A = \{1, 2, 3, 4, 5\}, B = \{4, 5, 6, 7, 8\}$$

and

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}. \quad A - B = \{1, 2, 3\}.$$  

$$A \cap B = \{4, 5\}. \quad B - A = \{6, 7, 8\}.$$  

$$\bar{A} = \{0, 6, 7, 8, 9, 10\}. \quad A \oplus B = \{1, 2, 3, 6, 7, 8\}$$  

$$\bar{B} = \{0, 1, 2, 3, 9, 10\}.$$
The following **set identities** correspond to the logical equivalences discussed in Lecture 2.

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
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<tbody>
<tr>
<td>( A \cap U = A )</td>
<td>Identity laws</td>
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<td>( A \cup \emptyset = A )</td>
<td>Domination laws</td>
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<td>( A \cup U = U )</td>
<td>Idempotency laws</td>
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<td>( A \cap \emptyset = \emptyset )</td>
<td>Inverse law</td>
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<td>( A \cup A = A )</td>
<td>Double complement law</td>
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<td>( A \cap A = A )</td>
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<td>( A \cup \overline{A} = U )</td>
<td>Associative laws</td>
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<td>( A \cap \overline{A} = \emptyset )</td>
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<td>( \overline{\overline{A}} = A )</td>
<td>DeMorgan’s laws</td>
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<td>( A \cup B = B \cup A )</td>
<td>Absorption law</td>
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<tr>
<td>( A \cap B = B \cap A )</td>
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<tr>
<td>( (A \cup B) \cup C = A \cup (B \cup C) )</td>
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<tr>
<td>( (A \cap B) \cap C = A \cap (B \cap C) )</td>
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<td>( A \bar{\cap} B = \bar{A} \cup \bar{B} )</td>
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<td>( A \bar{\cup} B = \bar{A} \cap \bar{B} )</td>
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<td>( A \cup (A \cap B) = A )</td>
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<td>( A \cap (A \cup B) = A )</td>
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Table 1: Algebra of Sets
Definition 7  The **union** of a collection of sets is the set that contains those elements that are members of at least one set in the collection:

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_1 \cup \ldots \cup A_n = \{x|\exists i(x \in A_i)\}. \]

Definition 8  The **intersection** of a collection of sets is the set that contains those elements that are members of all the sets in the collection:

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_1 \cap \ldots \cap A_n = \{x|\forall i(x \in A_i)\}. \]

These definitions can be applied to infinite collections of sets as well.

Definition 9  A collection of nonempty sets \( \{A_1, A_2, \ldots, A_n\} \) is a **partition** of a set \( A \) if and only if,

1. \( A = A_1 \cup A_2 \cup \cdots \cup A_n; \)
2. \( A_1, A_2, \ldots, A_n \) are mutually disjoint.

Example:  Let \( A = \{1, 2, 3, 4, 5, 6\} \). Then, \( A_1 = \{1, 2\}, A_2 = \{3, 4\}, \) and \( A_3 = \{5, 6\} \) is a partition of \( A \).

**Membership Table**

Set identities can also be proved using membership tables. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

**Example:**  Use a membership table to show that

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
### Sets operations

<table>
<thead>
<tr>
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<th>B</th>
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Generalized Union and Intersection

Example: Let

(i) If \( A_i = [i, \infty) \), \( 1 \leq i < \infty \), then:

\[
\bigcup_{i=1}^{n} A_i = [1, \infty).
\]

and

\[
\bigcup_{i=1}^{n} A_i = [n, \infty).
\]

(ii) If \([a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}\), then:

\[
\bigcup_{n=1}^{\infty} [n, n+1] = [1, 2] \cup [2, 3] \cup [3, 4] \cup ... = [1, \infty).
\]

• What is

\[
\bigcap_{n=1}^{\infty} \left[ -\frac{1}{n}, \frac{1}{n} \right]?
\]

• Computer representation of sets. Having discussed about sets and the various operations on them, it is pertinent to ask the question “what is an efficient way of representing sets on the computer?”

• It turns out that sets can be represented using bit strings which provides for easy computation of subsets, set complement, union, intersection and difference.

• The idea is that if a universal set has \( n \) elements, then its subsets etc, are represented as bit strings of length \( n \) as well.

• The bit string corresponding to a subset \( A \) of \( U \) has a 1 in position \( k \) if \( a_k \in A \), and has a 0 in this position if \( a_k \notin A \).
Representing Sets On the Computer

- E.g., the subset
  \[ A = \{2, 4, 6, 8, 10\} \]
  of
  \[ U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \]
  is represented as a bit string of length 10 as 0101010101 while
  \[ \bar{A} = \{1, 3, 5, 7, 9\} \]
  is represented as 1010101010.

- Furthermore, if
  \[ B = \{3, 6, 9\} \]
  represented as 0010010010, then
  \[ A \cap B = \{6\} \]
  is represented as 0000010000 while
  \[ B \cup \bar{A} = \{1, 3, 5, 6, 7, 9\} \]
  is represented as 1010111010.

- To obtain the bit string for the union and intersection of two sets, we perform bitwise Boolean operations on the bit strings representing the two sets.

- Another advantage of this representation for sets is in generating the combinations (subsets) of a set since, for example, the number of subsets of \(\{1, 2, \ldots, n\}\) is equivalent to the number of bit strings of length \(n\) (See Rosen, Section 4.6).

The cardinality of the union of sets

\[ \text{Definition 10} \quad \text{Let } A \text{ and } B \text{ be sets. Then,} \]
\[ |A \cup B| = |A| + |B| - |A \cap B|. \]  

\[ \text{Definition 11} \quad \text{Let } A, B \text{ and } C \text{ be sets. Then,} \]
\[ |A \cup B \cup C| = |A| + |B| + |C| \]
\[ - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \]